

INTEGRABLE COMPLEX STRUCTURES ON TWISTOR SPACES

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CONTENTS

1. Introduction	1
2. Complex Structures on Twistor Spaces	5
2.1. Preliminaries	5
2.2. Horizontal Distributions and Splittings	6
2.3. The Complex Structures	8
2.4. (1,1) Curvature	11
2.5. Other Curvature Conditions	13
3. More Examples	15
3.1. Chern Connections	15
3.2. $\bar{\partial}$ -operators	16
3.3. Three Forms	17
3.4. Holomorphic Sections of Twistor space	19
4. Twistors and Grassmannians	24
4.1. Embedding the Fibers	24
4.2. The Holomorphic Embedding	25
4.3. Proof of Proposition 4.8	27
4.4. Corollaries of the Embedding	28
5. Acknowledgments	31
References	31

1. INTRODUCTION

In the 1970's, Atiyah, Hitchin and Singer introduced a tautological almost complex structure on a certain twistor space that along with its generalizations have had, until today, a major impact on differential and complex geometry [3, 25]. The twistor space that they considered was $\mathcal{T}^+(TM)$, the bundle of complex structures fibered over an oriented Riemannian four manifold that are compatible with the metric and orientation; the almost complex structure was J_{taut}^{∇} , where ∇ is the Levi Civita connection, and is defined in Section 2.3. The importance of J_{taut}^{∇} was found to lie in the times when it was integrable, which was when the four manifold was anti-selfdual, and one of its many applications was the construction of instantons on S^4 [2].

Given its success in four dimensions, J_{taut}^∇ was generalized in [4, 24] to the twistor space $\mathcal{C}(TM) = \{J \in \text{End}TM \mid J^2 = -1\}$, where M is any even dimensional manifold and ∇ is any connection. Its integrability conditions were then explored with the hope that J_{taut}^∇ would again lead to major results about the base manifold. However, it was found that these conditions imposed severe restrictions on the curvature of the connection and in almost all cases J_{taut}^∇ was not integrable, thus limiting its applications in differential geometry.

Of course this did not prevent mathematicians from taking advantage of the rare times when it was integrable on either $\mathcal{C}(TM)$ or on submanifolds within, and applying J_{taut}^∇ to advance, for example, the theory of harmonic mappings, integrable systems and hyperkahler geometry. With all of its successes, however, the fact remains that the rarity of the integrability of J_{taut}^∇ has greatly hindered its use in deriving results about the geometry of the base manifold M in higher dimensions. And it is natural to wonder whether there exist other almost complex structures on twistor spaces whose integrability conditions are more easily satisfied –especially in every dimension– and at the same time can be used to derive results about the base manifold.

The purpose of this paper is to demonstrate that such almost complex structures do indeed exist if we assume that M is itself equipped with a complex structure I . Whereas \mathcal{J}_{taut} , which will stand for J_{taut}^∇ for an unspecified connection, can only be defined on twistor spaces that are associated to TM , the almost complex structures that we introduce in Section 2.3 are defined on more general twistor spaces that are associated to any even dimensional real vector bundle. Denoting such a bundle by E , in that section, we define the almost complex structure $\mathcal{J}^{(\nabla, I)}$ on $\mathcal{C}(E) = \{J \in \text{End}E \mid J^2 = -1\}$; it depends on a choice of a connection ∇ on E , similar to the definition of \mathcal{J}_{taut} . However, unlike \mathcal{J}_{taut} , the conditions on the connection ∇ for $\mathcal{J}^{(\nabla, I)}$ to be integrable are easily fulfilled. By computing its Nijenhuis tensor, we prove in Theorem 2.17, that $\mathcal{J}^{(\nabla, I)}$ on $\mathcal{C}(E)$ is automatically integrable if the curvature of ∇ , R^∇ , is (1,1) with respect to I , i.e., $R^\nabla(I\cdot, I\cdot) = R^\nabla(\cdot, \cdot)$. Moreover if g is a metric on E and $\nabla g = 0$ then $\mathcal{J}^{(\nabla, I)}$ on $\mathcal{T}(E, g) = \{J \in \mathcal{C}(E) \mid g(J\cdot, J\cdot) = g(\cdot, \cdot)\}$ is integrable if and only if R^∇ is (1,1) (Proposition 2.28). Under these conditions, the projection map $\pi : (\mathcal{C}(E), \mathcal{J}^{(\nabla, I)}) \rightarrow (M, I)$ becomes a holomorphic submersion, a property that would never hold if we were to replace $\mathcal{J}^{(\nabla, I)}$ by \mathcal{J}_{taut} .

While we will use $(\mathcal{C}(E), \mathcal{J}^{(\nabla, I)})$ to derive results about the base manifold (M, I) in a forthcoming paper [13] (see also below), the focus of our present paper is to describe various examples of vector bundles that admit connections with (1,1) curvature and the resulting holomorphic structures on the twistor spaces. For instance, in Section 3.2 we demonstrate that a general holomorphic Hermitian bundle, (E, g, J) , admits many such connections. The Chern connection is of course an example, but as we show, $\bar{\partial}$ closed sections $D \in \Gamma(T^{*0,1} \otimes \wedge^2 E^{*1,0})$ and $\alpha \in \Gamma(T^{*0,1} \otimes \wedge^2 E^{1,0})$ can also be used

to define connections on E with $(1,1)$ curvature. In the case when $E = TM$, D is a $\bar{\partial}$ closed $(2,1)$ form and our goal in Section 3.3 is to describe how such forms naturally appear on well known classes of Hermitian manifolds. For example, SKT manifolds, bihermitian manifolds –also known as generalized Kahler manifolds– as well as Calabi-Yau manifolds [9, 11, 1, 16] all admit $\bar{\partial}$ closed $(2,1)$ forms and thus complex structures on their twistor spaces.

Another example of a connection whose curvature is $(1,1)$ is the Levi Civita connection, ∇ , on the tangent bundle of any anti-selfdual Hermitian four manifold, (M, g, I) , whose orientation is determined by I . In this case, there are two integrable complex structures on $\mathcal{T}^+(TM)$, which is the subbundle of $\mathcal{T}(TM)$ whose elements induce the same orientation as that of I . The first is $\mathcal{J}^{(\nabla, I)}$, as it is integrable on all of $\mathcal{T}(TM)$, and the second is $\mathcal{J}_{taut}^\nabla$, as was discussed above. In Example 2.23 we consider some of their interaction as well as several bundles fibered over $\mathcal{T}^+(TM)$ that themselves admit connections with $(1,1)$ curvature with respect to both $\mathcal{J}^{(\nabla, I)}$ and $\mathcal{J}_{taut}^\nabla$.

To obtain even more examples, in Section 3.4 we consider a bundle $E \rightarrow (M, I)$ that is already equipped with a connection ∇ with $(1,1)$ curvature and show how to use holomorphic sections of the corresponding twistor space $(\mathcal{C}(E), \mathcal{J}^{(\nabla, I)}) \rightarrow (M, I)$ to produce other connections on E that satisfy the same curvature condition– and hence produce other complex structures on $\mathcal{C}(E)$. For example, by Proposition 3.13, if J is a holomorphic section then $\nabla' = \nabla + \frac{1}{2}(\nabla J)J$ is another connection that has $(1,1)$ curvature. This connection also satisfies $\nabla' J = 0$ and will be used especially in [13] to derive results about the base manifold (M, I) (see also below).

We then provide in Section 3.4.2 several examples of holomorphic sections of twistor spaces. For instance, the twistor spaces of the manifolds already considered above– SKT, bihermitian and Calabi-Yau– all admit holomorphic sections. In addition, there are certain twistor spaces that are fibered over other twistor spaces that admit holomorphic sections. More specifically, if we consider $\pi : (\mathcal{C}(E), \mathcal{J}^{(\nabla, I)}) \rightarrow (M, I)$ then $\phi \in \Gamma(\pi^* \text{End} E)$ defined by $\phi|_K = K$ is a holomorphic section of $(\mathcal{C}(\pi^* E), \mathcal{J}^{(\pi^* \nabla, \mathcal{I})}) \rightarrow (\mathcal{C}(E), \mathcal{I})$, where $\mathcal{I} = \mathcal{J}^{(\nabla, I)}$. Moreover, in the case when $E = TM$, $\mathcal{J}_{taut}^\nabla$ – although almost never integrable– is always a holomorphic section of $(\mathcal{C}(TC), \mathcal{J}) \rightarrow (\mathcal{C}, \mathcal{J}^{(\nabla, I)})$, where $\mathcal{C} := \mathcal{C}(TM)$ and \mathcal{J} is some appropriate complex structure to be defined. More examples of holomorphic sections as well as their associated connections with $(1,1)$ curvature are also discussed.

If we now consider a bundle $E \rightarrow (M, I)$ with a connection ∇ that has $(1,1)$ curvature then one may wonder whether $(\mathcal{C}(E), \mathcal{J}^{(\nabla, I)})$ can be holomorphically embedded into a more familiar complex manifold. The key in finding a suitable manifold, is to notice the fact that if we \mathbb{C} -linearly extend ∇ to a complex connection on $E_{\mathbb{C}} := E \otimes_{\mathbb{R}} \mathbb{C}$ then R^∇ is $(1,1)$ if and only if $\nabla^{0,1}$ is a $\bar{\partial}$ -operator on $E_{\mathbb{C}}$. Hence given ∇ on E with $(1,1)$ curvature, we have two associated complex analytic manifolds: the first

is $(\mathcal{C}(E), \mathcal{J}^{(\nabla, I)})$ and the other is the holomorphic Grassmannian bundle $Gr_n(E_{\mathbb{C}})$ ($\dim_{\mathbb{R}} E = 2n$), and in Section 4 we holomorphically embed $\mathcal{C}(E)$ into this latter bundle. By then considering $\mathcal{C}(E)$ as a complex submanifold of $Gr_n(E_{\mathbb{C}})$, we derive a number of corollaries about the holomorphic structure of twistor spaces. For example, we derive conditions on two connections ∇ and ∇' that are defined on $E \rightarrow (M, I)$ with $(1,1)$ curvature, so that the twistor spaces $(\mathcal{C}(E), \mathcal{J}^{(\nabla, I)})$ and $(\mathcal{C}(E), \mathcal{J}^{(\nabla', I)})$ are equivalent under a fiberwise biholomorphism. We then use this to prove that certain complex structures that we defined in Section 3.2 on the twistor spaces associated to Hermitian bundles are in fact biholomorphic. Other corollaries of the holomorphic embedding are given in Sections 4.4.1 and 4.4.2.

Having in this paper defined holomorphic embeddings, given examples, and explored different properties of $(\mathcal{C}(E), \mathcal{J}^{(\nabla, I)}) \rightarrow (M, I)$, the goal of our forthcoming paper [13] will be to use these holomorphic twistor spaces to derive results about the complex geometry of the base manifold (M, I) . Indeed, given the standard setup of $E \rightarrow (M, I)$ with a connection ∇ such that R^{∇} is $(1,1)$, we will demonstrate how to use holomorphic sections of $(\mathcal{C}(E), \mathcal{J}^{(\nabla, I)}) \rightarrow (M, I)$ together with the other associated connections with $(1,1)$ curvature that were described above, to decompose the complex manifold (M, I) into various holomorphic subvarieties.

For example, by considering holomorphic sections of twistor spaces fibered over a bihermitian manifold we will construct holomorphic subvarieties in the manifold that are new to the literature. The importance of these subvarieties lies in their connection to the known holomorphic and real Poisson structures on the manifold. In fact, some of these subvarieties refine the structure of the degeneracy loci of the Poisson structures and our objective is to introduce new tools to study these loci via twistor spaces. Please see [13] for the details.

As another example, for $\mathcal{I} = \mathcal{J}^{(\nabla, I)}$, we will show in [13] that the holomorphic section $\phi : (\mathcal{C}(E), \mathcal{I}) \rightarrow (\mathcal{C}(\pi^*E), \mathcal{J}^{(\pi^*\nabla, \mathcal{I})})$, which was defined previously, together with other sections induce several stratifications of $(\mathcal{C}(E), \mathcal{I})$ whose strata are complex submanifolds (see also Example 2.29). These strata are defined fiberwise and their fibers correspond to certain Schubert cells. As $\mathcal{J}_{taut}^{\nabla}$ is closely related to ϕ in the case when $E = TM$ (see Definition 2.12), it too will induce stratifications of the twistor space $\mathcal{C}(TM)$. Hence, for these applications, we will be focusing on exploiting $\mathcal{J}_{taut}^{\nabla}$'s holomorphicity property, which always holds true, as opposed to its integrability property, which almost never does.

These stratifications not only provide examples of the holomorphic subvarieties that are induced from twistor space, but are also important in deriving general results about them. Indeed, we will show in [13] that these subvarieties can be viewed as the intersections of the closures of the different

holomorphic strata in twistor space. From this viewpoint we will, in particular, derive lower bounds on the dimensions of the subvarieties and find conditions that are necessary for their existence.

Further applications of holomorphic twistor spaces will be described in [13] as well as in other forthcoming papers. At the present time, let us turn to the task of defining integrable complex structures on twistor spaces. We begin with the following preliminaries.

2. COMPLEX STRUCTURES ON TWISTOR SPACES

2.1. Preliminaries. Let V be a $2n$ dimensional real vector space and let $\mathcal{C}(V) = \{J \in \text{End}V \mid J^2 = -1\}$ be one of its twistor spaces. To describe some of the properties of $\mathcal{C}(V)$, consider the action of $GL(V)$ on $\text{End}V$ via conjugation: $B \cdot A = BAB^{-1}$. As $\mathcal{C}(V)$ is a particular orbit of this action, it is isomorphic to

$$GL(V)/GL(V, I),$$

where $I \in \mathcal{C}(V)$ and $GL(V, I) = \{B \in GL(V) \mid [B, I] = 0\} \cong GL(n, \mathbb{C})$. It then follows that the dimension of $\mathcal{C}(V)$ is $2n^2$ and that if we consider $\mathcal{C}(V)$ as a submanifold of $\text{End}V$ then

$$T_J\mathcal{C} = [\text{End}V, J] = \{A \in \text{End}V \mid \{A, J\} = 0\}.$$

With this, we may define a natural almost complex structure on $\mathcal{C}(V)$ that is well known to be integrable:

$$I_{\mathcal{C}}A = JA, \text{ for } A \in T_J\mathcal{C}.$$

If we now equip V with a positive definite metric g then another twistor space that we will consider is $\mathcal{T}(V, g) = \{J \in \mathcal{C}(V) \mid g(J\cdot, J\cdot) = g(\cdot, \cdot)\}$. In this case, \mathcal{T} is an orbit of the action of $O(V, g)$ on $\text{End}V$ by conjugation, and is thus isomorphic to the Hermitian symmetric space $O(V, g)/U(I)$, where $I \in \mathcal{T}$ and $U(I) \cong U(n)$. It then follows that the dimension of \mathcal{T} is $n(n-1)$ and that if we consider \mathcal{T} as a submanifold of $\text{End}V$ then

$$T_J\mathcal{T} = [\mathfrak{o}(V, g), J] = \{A \in \mathfrak{o}(V, g) \mid \{A, J\} = 0\}.$$

As $I_{\mathcal{C}}$ naturally restricts to $T_J\mathcal{T}$, \mathcal{T} is a complex submanifold of \mathcal{C} .

We will show in [13] that \mathcal{C} admits other natural complex submanifolds that form the strata of several stratifications of both \mathcal{T} and \mathcal{C} .

2.1.1. Twistors of Bundles. Let now $E \rightarrow M$ be an even dimensional vector bundle fibered over an even dimensional manifold. Generalizing the previous discussion to vector bundles, we will define $\mathcal{C}(E) = \{J \in \text{End}E \mid J^2 = -1\}$, which is a fiber subbundle of the total space of $\pi : \text{End}E \rightarrow M$ with general fiber $\mathcal{C}(E_x)$, for $x \in M$. Since the fibers of $\pi_{\mathcal{C}} : \mathcal{C}(E) \rightarrow M$ are complex manifolds, $\mathcal{C}(E)$ naturally admits the complex vertical distribution $VC \subset TC(E)$, where $V_J\mathcal{C} = T_J\mathcal{C}(E_{\pi(J)}) \cong [\text{End}E|_{\pi(J)}, J]$. Using the section $\phi \in \Gamma(\pi_{\mathcal{C}}^*\text{End}E)$ defined by $\phi|_J = J$, we will then identify VC with the subbundle $[\pi_{\mathcal{C}}^*\text{End}E, \phi]$ of $\pi_{\mathcal{C}}^*\text{End}E$.

Letting g be a positive definite metric on E , we will also consider $\mathcal{T}(E, g) = \{J \in \mathcal{C}(E) \mid g(J\cdot, J\cdot) = g(\cdot, \cdot)\}$. Similar to the case of $\mathcal{C}(E)$, $\mathcal{T}(E, g)$ naturally admits the complex vertical distribution $V\mathcal{T}$, defined by $V_J\mathcal{T} = T_J\mathcal{T}(E_{\pi(J)}) \cong [\mathfrak{o}(E_{\pi(J)}, g), J]$. If we denote the projection map from $\mathcal{T}(E, g)$ to M by $\pi_{\mathcal{T}}$ then we will identify $V\mathcal{T}$ with the subbundle $[\pi_{\mathcal{T}}^*\mathfrak{o}(E, g), \phi]$ of $\pi_{\mathcal{T}}^*EndE$, where now $\phi \in \Gamma(\pi_{\mathcal{T}}^*EndE)$.

Notation 2.1. *As was done above and will be continued below, we will at times denote $\mathcal{C}(E)$ by \mathcal{C} and $\mathcal{T}(E, g)$ by $\mathcal{T}(E)$, $\mathcal{T}(g)$ or just \mathcal{T} . Moreover, there are also times when we will denote $\pi_{\mathcal{C}}$ or $\pi_{\mathcal{T}}$ by just π .*

2.2. Horizontal Distributions and Splittings. With this background at hand, we will now take the first steps in defining integrable complex structures on $\mathcal{C}(E)$ and $\mathcal{T}(E, g)$ in the case when M is a complex manifold. Given a connection ∇ on E we will define the horizontal distribution $H^\nabla\mathcal{C}$ in $T\mathcal{C}$, so that this latter bundle splits into $V\mathcal{C} \oplus H^\nabla\mathcal{C}$. Similarly, in the case when g is a metric on E and ∇ is a metric connection, we will describe how to split $T\mathcal{T}$ into $V\mathcal{T} \oplus H^\nabla\mathcal{T}$. Once we have described these splittings we will define the desired complex structures on the above twistor spaces in Section 2.3.

To begin, let, as above, $E \rightarrow M$ be a vector bundle, though the base manifold is not yet assumed to be a complex manifold, and let ∇ be any connection. As \mathcal{C} is a fiber subbundle of the total space of $\pi : EndE \rightarrow M$, we will find it convenient to split its tangent bundle by first splitting $TEndE$.

Although there are other ways to define this splitting the basic idea here is to use parallel translation with respect to ∇ . First, if $A \in EndE$ and $\gamma : \mathbb{R} \rightarrow M$ satisfies $\gamma(0) = \pi(A)$ then the parallel translate of A along γ will be denoted by $A(t)$. The horizontal distribution $H^\nabla EndE$ in $TEndE$ is then defined as follows.

Definition 2.2. Let $H_A^\nabla EndE = \{\frac{dA(t)}{dt}|_{t=0} \mid \text{for all } \gamma, \gamma(0) = \pi(A)\}$.

It is straightforward to show that $H^\nabla EndE$ is a complement to the vertical distribution:

Lemma 2.3. $TEndE = VEndE \oplus H^\nabla EndE$.

Remark 2.4. *The above procedure can actually be used to split the tangent bundle of any vector bundle with a connection. Another way to define such a splitting is to consider the bundle as associated to its frame bundle and then use the standard theory of connections. These two methods yield the same splittings and are essentially equivalent.*

Now if $J \in \mathcal{C} \subset EndE$ and $\gamma : \mathbb{R} \rightarrow M$ is a curve that satisfies $\gamma(0) = \pi(J)$ then it is clear that the associated parallel translate $J(t)$ lies in \mathcal{C} for all relevant $t \in \mathbb{R}$. It then follows that $H_J^\nabla EndE$ lies in $T_J\mathcal{C}$, so that we have:

Lemma 2.5. $T_J\mathcal{C} = V_J\mathcal{C} \oplus H_J^\nabla\mathcal{C}$, where $H_J^\nabla\mathcal{C} = H_J^\nabla\text{End}E$.

Similarly, if g is a metric on E and $\nabla g = 0$ then the parallel translate of $J \in \mathcal{T}$ along γ lies in \mathcal{T} . We thus have

Lemma 2.6. $T_J\mathcal{T} = V_J\mathcal{T} \oplus H_J^\nabla\mathcal{T}$, where $H_J^\nabla\mathcal{T} = H_J^\nabla\text{End}E$.

With the above splittings, it will be useful for later calculations to derive a certain formula for the vertical projection operator $P^\nabla : T\text{End}E \rightarrow V\text{End}E \cong \pi^*\text{End}E$, which, upon suitable restriction, will also be valid for the corresponding projection operators for $T\mathcal{C}$ and $T\mathcal{T}$. The formula will depend on the tautological section ϕ of $\pi^*\text{End}E$ that is defined by $\phi|_A = A$:

Proposition 2.7. Let $X \in T_A\text{End}E$, then

$$P^\nabla(X) = (\pi^*\nabla)_X\phi,$$

where we are considering P^∇ to be a section of $T^*\text{End}E \otimes \pi^*\text{End}E$.

Proof of Proposition 2.7. Let $\{e_i\}$ be a local frame for E over some open set $U \subset M$ about the point $\pi(A)$, where $A \in \text{End}E$, and let $\{e_i \otimes e^j\}$ be the corresponding frame for $\text{End}E$. Then for $X \in T_A\text{End}E$,

$$(2.1) \quad (\pi^*\nabla)_X\phi = (\pi^*\nabla)_X\phi_j^i\pi^*(e_i \otimes e^j)$$

$$(2.2) \quad = d\phi_j^i(X)e_i \otimes e^j|_{\pi(A)} + A_j^i\nabla_{\pi_*X}e_i \otimes e^j.$$

Let us now consider the following two cases.

(A) Let X be an element of $V_A\text{End}E$, which for the moment is not identified with $\text{End}E|_{\pi(A)}$, so that $\pi_*X = 0$. Also let $A(t)$ be a curve in $\text{End}E|_{\pi(A)}$ such that $A(0) = A$ and $\frac{dA(t)}{dt}|_{t=0} = X$. Then by Equation 2.2, $(\pi^*\nabla)_X\phi = \frac{dA(t)_j^i}{dt}|_{t=0}e_i \otimes e^j|_{\pi(A)} = P^\nabla(X) \in \text{End}E|_{\pi(A)}$.

B) Let $X \in H_A^\nabla\text{End}E$ so that it equals $\frac{d}{dt}A(t)|_{t=0}$, where $A(t)$ is the parallel translate of A along some curve $\gamma : \mathbb{R} \rightarrow M$ that satisfies $\gamma(0) = \pi(A)$. As $d\phi_j^i(X) = \frac{d}{dt}A(t)_j^i|_{t=0}$, Equation 2.2 becomes $\frac{d}{dt}A(t)_j^i|_{t=0}e_i \otimes e^j|_{\pi(A)} + A_j^i\nabla_{\frac{d\gamma}{dt}|_{t=0}}e_i \otimes e^j$, which is zero since $A(t)$ is parallel. \square

If we consider the corresponding projection operator $P^\nabla : T\mathcal{C} \rightarrow V\mathcal{C}$ then it follows from the above proposition that $P^\nabla(X) = (\pi_C^*\nabla)_X\phi$, where ϕ is now a section of $\pi_C^*\text{End}E \rightarrow \mathcal{C}$. Note that since $\phi^2 = -1$, $(\pi_C^*\nabla)_X\phi$, for $X \in T_J\mathcal{C}$, is indeed contained in $V_J\mathcal{C} = \{A \in \text{End}E|_{\pi(J)} | \{A, J\} = 0\}$. In the case when g is a metric on E and $\nabla g = 0$, an analogous formula holds for $P^\nabla : T\mathcal{T} \rightarrow V\mathcal{T}$.

Remark 2.8. We respectfully report that similar formulas for the projection operators for $T\mathcal{C}$ and $T\mathcal{T}$ were derived in [24] but with a small error.

2.3. The Complex Structures. Now let $E \longrightarrow (M, I)$ be an even dimensional bundle that is fibered over an almost complex manifold and let ∇ be a connection on E . We will define the following almost complex structure on the total space of $\pi : \mathcal{C}(E) \longrightarrow M$ and explore its integrability conditions in the next section.

Definition 2.9. $\mathcal{J}^{(\nabla, I)}$: First use ∇ to split

$$T\mathcal{C} = V\mathcal{C} \oplus H^\nabla\mathcal{C},$$

and then let

$$\begin{aligned} (1) \quad & \mathcal{J}^{(\nabla, I)}A = JA \\ (2) \quad & \mathcal{J}^{(\nabla, I)}v^\nabla = (Iv)^\nabla, \end{aligned}$$

where $A \in V_J\mathcal{C} \subset \text{End}E|_{\pi(J)}$ and $v^\nabla \in H_J^\nabla\mathcal{C}$ is the horizontal lift of $v \in T_{\pi(J)}M$.

In other words, $\mathcal{J}^{(\nabla, I)}$ on $V\mathcal{C} \oplus H^\nabla\mathcal{C}$ equals $\phi \oplus \pi^*I$, where we have identified $V\mathcal{C}$ with $[\pi^*\text{End}E, \phi]$ and $H^\nabla\mathcal{C}$ with π^*TM .

It then follows from the definition of $\mathcal{J}^{(\nabla, I)}$ that π is pseudoholomorphic:

Proposition 2.10. $\pi : (\mathcal{C}, \mathcal{J}^{(\nabla, I)}) \longrightarrow (M, I)$ is a pseudoholomorphic submersion.

In the case when g is a metric on E and ∇ is a metric connection, the claim is that $\mathcal{J}^{(\nabla, I)}$ on \mathcal{C} restricts to \mathcal{T} , so that $\mathcal{T} \subset (\mathcal{C}, \mathcal{J}^{(\nabla, I)})$ is an almost complex submanifold. The reason is that $T_J\mathcal{T}$ splits into $V_J\mathcal{T} \oplus H_J^\nabla\mathcal{T}$, where $H_J^\nabla\mathcal{T} = H_J^\nabla\mathcal{C} = H_J^\nabla\text{End}E$, as explained in the previous section.

Remark 2.11. It should be noted that $\mathcal{J}^{(\nabla, I)}$ has not yet been studied in this generality in the literature. In [28], Vaisman did study $\mathcal{J}^{(\nabla, I)}$ only in the special case when $E = TM$ and $\nabla I = 0$ and only on certain submanifolds of $\mathcal{C}(TM)$. However, for our applications we do not want to restrict ourselves to $E = TM$ and we especially do not want to require $\nabla I = 0$.

With $\mathcal{J}^{(\nabla, I)}$ defined, let us now compare it to the tautological almost complex structures on twistor spaces that are usually considered in the literature [3, 24, 4]. If ∇' is a connection on $TM \longrightarrow M$, where here M is any even dimensional manifold, then based on the splitting of $T\mathcal{C}$ into $V\mathcal{C} \oplus H^{\nabla'}\mathcal{C}$, we define $\mathcal{J}_{\text{taut}}^{\nabla'}$ on $\mathcal{C}(TM)$ as follows.

Definition 2.12. Let $\mathcal{J}_{\text{taut}}^{\nabla'} = \phi \oplus \phi$, where we have identified $V\mathcal{C}$ with $[\pi^*\text{End}TM, \phi]$ and $H^{\nabla'}\mathcal{C}$ with π^*TM , and where the first ϕ factor acts by left multiplication.

To compare it to $\mathcal{J}^{(\nabla, I)}$, note that $\mathcal{J}_{\text{taut}}^{\nabla'}$ does not require M to admit an almost complex structure, while the former one does. On the other hand, $\mathcal{J}_{\text{taut}}^{\nabla'}$ is only defined for the bundle $E = TM$ whereas $\mathcal{J}^{(\nabla, I)}$ is defined for any even dimensional vector bundle. Also, given (M, I) , the projection map $(\mathcal{C}(TM), \mathcal{J}_{\text{taut}}^{\nabla'}) \longrightarrow (M, I)$ is never pseudoholomorphic, whereas

$(\mathcal{C}(E), \mathcal{J}^{(\nabla, I)}) \longrightarrow (M, I)$ is always so. Lastly, $\mathcal{J}_{taut}^{\nabla'}$ is rarely integrable—except in special cases such as when M is an anti-selfdual four manifold, as explained in the introduction [3]—whereas the integrability conditions of $\mathcal{J}^{(\nabla, I)}$ are very natural to be fulfilled, as we will show below.

Although $\mathcal{J}^{(\nabla, I)}$ and $\mathcal{J}_{taut}^{\nabla'}$ are defined quite differently and have opposite properties, they are still related holomorphically. Indeed, given (M, I) and a connection ∇ on TM and letting $\mathcal{C} = \mathcal{C}(TM)$, we will show in Section 3.4.2 that $\mathcal{J}_{taut}^{\nabla}$, which is almost never integrable, is always a pseudoholomorphic section of $(\mathcal{C}(TC), \mathcal{J}) \longrightarrow (\mathcal{C}, \mathcal{J}^{(\nabla, I)})$ for some appropriately chosen \mathcal{J} . We will also explore more of their interaction in [13].

Having compared the above almost complex structures, let us now return to the general setup of a vector bundle $E \longrightarrow (M, I)$ that is fibered over an almost complex manifold and that is equipped with a connection ∇ . The goal is to determine the conditions on I and the curvature of ∇ , R^∇ , that are equivalent to the integrability of $\mathcal{J}^{(\nabla, I)}$ not just on \mathcal{C} but on other almost complex submanifolds \mathcal{C}' as well. Although these conditions can be worked out for any \mathcal{C}' , we will focus on the case when the corresponding projection map $\pi_{\mathcal{C}'} : \mathcal{C}' \longrightarrow M$ is a surjective submersion. If g is a metric on E and $\nabla g = 0$ then as an example we can take $\mathcal{C}' = \mathcal{T}(g)$. We will describe other examples below in Section 2.5 and especially in [13].

The method that we will use to explore the integrability conditions of $\mathcal{J}^{(\nabla, I)}$ on \mathcal{C}' is to calculate its Nijenhuis tensor on \mathcal{C} .

2.3.1. Nijenhuis Tensor. In this section, let $\pi : \mathcal{C}(E) \longrightarrow M$ be the projection map and define $\mathcal{J} := \mathcal{J}^{(\nabla, I)}$ and $P := P^\nabla : TC \longrightarrow VC \subset \pi^* \text{End} E$, as in Section 2.2. We will presently compute the Nijenhuis tensor, $N^\mathcal{J}$, of \mathcal{J} that is given by

$$N^\mathcal{J}(X, Y) = [\mathcal{J}X, \mathcal{J}Y] - \mathcal{J}[\mathcal{J}X, Y] - \mathcal{J}[X, \mathcal{J}Y] - [X, Y],$$

in terms of the Nijenhuis tensor of I and the curvature of ∇ , R^∇ .

Proposition 2.13. *Let $X, Y \in T_J \mathcal{C}$ and let $v = \pi_* X$ and $w = \pi_* Y$. Then*

- 1) $\pi_* N^\mathcal{J}(X, Y) = N^I(v, w)$
- 2) $P N^\mathcal{J}(X, Y) = [R^\nabla(v, w) - R^\nabla(Iv, Iw), J] + J[R^\nabla(Iv, w) + R^\nabla(v, Iw), J]$.

Proof of Proposition 2.13, Part 1). This easily follows from the fact that if $X \in \Gamma(TC)$ is π -related to $v \in \Gamma(TM)$ then $\mathcal{J}X$ is π -related to Iv . \square

Letting, as above, $\phi \in \Gamma(\pi^* \text{End} E)$ be defined by $\phi|_J = J$, the proof of Part 2 of the proposition, will be based on the following lemma.

Lemma 2.14. *Let $X, Y \in \Gamma(TC)$. Then*

$$P^\nabla([X, Y]) = -[R^{\pi^* \nabla}(X, Y), \phi] + \pi^* \nabla_X P(Y) - \pi^* \nabla_Y P(X).$$

Proof. Consider

$$\begin{aligned} P^\nabla([X, Y]) &= \pi^* \nabla_{[X, Y]} \phi \\ &= -R^{(\pi^* \nabla, \pi^* \text{End} E)}(X, Y) \phi + \pi^* \nabla_X \pi^* \nabla_Y \phi - \pi^* \nabla_Y \pi^* \nabla_X \phi, \end{aligned}$$

where $R^{(\pi^* \nabla, \pi^* \text{End} E)}$ is the curvature of $\pi^* \nabla$, which is considered as a connection on $\pi^* \text{End} E$. The lemma then follows from the identity:

$$R^{(\pi^* \nabla, \pi^* \text{End} E)}(X, Y) \phi = [R^{\pi^* \nabla}(X, Y), \phi].$$

□

Proof of Proposition 2.13, Part 2). Let $X, Y \in \Gamma(T\mathcal{C})$ and consider $PN^\mathcal{J}(X, Y) = P([\mathcal{J}X, \mathcal{J}Y] - \mathcal{J}[\mathcal{J}X, Y] - \mathcal{J}[X, \mathcal{J}Y] - [X, Y])$. By using the previous lemma as well as the fact that $P\mathcal{J} = \phi P$, we can express $PN^\mathcal{J}(X, Y)$ as the sum of two sets of terms. The first set involves the curvature of $\pi^* \nabla$:

$$[R^{\pi^* \nabla}(X, Y) - R^{\pi^* \nabla}(\mathcal{J}X, \mathcal{J}Y), \phi] + \phi[R^{\pi^* \nabla}(\mathcal{J}X, Y) + R^{\pi^* \nabla}(X, \mathcal{J}Y), \phi].$$

When restricted to $J \in \mathcal{C}$ this gives the expression for $PN^\mathcal{J}(X, Y)$ that is contained in Part 2 of the proposition.

The second set of terms is

$$\pi^* \nabla_{\mathcal{J}X} P(\mathcal{J}Y) - \phi \pi^* \nabla_{\mathcal{J}X} P(Y) - \phi \pi^* \nabla_X P(\mathcal{J}Y) - \pi^* \nabla_X P(Y) - (X \leftrightarrow Y).$$

Using $P\mathcal{J} = \phi P$, it easily follows that the first four terms and the last four, which are represented by $(X \leftrightarrow Y)$, separately add to zero. □

2.3.2. Integrability Conditions. We are now prepared to explore the integrability conditions of $\mathcal{J}^{(\nabla, I)}$ on \mathcal{C}' , where, as above, \mathcal{C}' is any almost complex submanifold of $(\mathcal{C}(E), \mathcal{J}^{(\nabla, I)})$ such that $\pi_{\mathcal{C}'} : \mathcal{C}' \rightarrow M$ is a surjective submersion. As is well known, $\mathcal{J}^{(\nabla, I)}$ on \mathcal{C}' will be integrable if and only if $\pi_* N^\mathcal{J}(X, Y)$ and $PN^\mathcal{J}(X, Y)$ are both zero $\forall X, Y \in T_J \mathcal{C}'$ and $\forall J \in \mathcal{C}'$. By Proposition 2.13, the first condition is equivalent to the vanishing of the Nijenhuis tensor of I , while the second is equivalent to

$$[R^\nabla(v, w) - R^\nabla(Iv, Iw), J] + J[R^\nabla(Iv, w) + R^\nabla(v, Iw), J] = 0$$

$\forall v, w \in T_{\pi(J)} M$ and $\forall J \in \mathcal{C}'$. To analyze this condition, we will express it in terms of $R^{0,2}$, the (0,2)-form part of the curvature R^∇ :

Lemma 2.15. *The condition*

$$[R^\nabla(v, w) - R^\nabla(Iv, Iw), J] + J[R^\nabla(Iv, w) + R^\nabla(v, Iw), J] = 0$$

$\forall v, w \in T_{\pi(J)} M$ holds true if and only if

$$[R^{0,2}, J]E_J^{0,1} = 0.$$

We thus have:

Theorem 2.16. *$(\mathcal{C}', \mathcal{J}^{(\nabla, I)})$ is a complex manifold if and only if*

- 1) I is integrable
- 2) $[R^{0,2}, J]E_J^{0,1} = 0, \quad \forall J \in \mathcal{C}'$.

Note that the second condition in the above theorem is equivalent to $R^{0,2} : E_J^{0,1} \longrightarrow E_J^{0,1}, \forall J \in \mathcal{C}'$.

2.4. (1,1) Curvature. Assuming henceforth that I is integrable, an important case of Part 2 of the above theorem that guarantees that $(\mathcal{C}', \mathcal{J}^{(\nabla, I)})$ is a complex manifold is when $R^{(0,2)} = 0$, or equivalently, when R^∇ is (1,1) with respect to I . In particular, we have:

Theorem 2.17. *Let $E \longrightarrow (M, I)$ be fibered over a complex manifold and let ∇ be a connection on E that has (1,1) curvature. Then $\mathcal{J}^{(\nabla, I)}$ is an integrable complex structure on $\mathcal{C}(E)$. In addition, if g is a metric on E and $\nabla g = 0$ then $\mathcal{T}(E, g)$ is a complex submanifold of $(\mathcal{C}(E), \mathcal{J}^{(\nabla, I)})$.*

If we \mathbb{C} -linearly extend ∇ to a complex connection on $E_{\mathbb{C}} := E \otimes_{\mathbb{R}} \mathbb{C}$ then the condition that R^∇ is (1,1) can also be expressed as $(\nabla^{0,1})^2 = 0$. We thus have:

Lemma 2.18. *Let ∇ be a connection on $E \longrightarrow (M, I)$. Then R^∇ is (1,1) if and only if $\nabla^{0,1}$ is a $\bar{\partial}$ -operator on $E_{\mathbb{C}}$.*

In Section 4.2 we will use the fact that $\nabla^{0,1}$ is a $\bar{\partial}$ -operator to holomorphically embed $(\mathcal{C}', \mathcal{J}^{(\nabla, I)})$ into a more familiar complex manifold that is associated to the holomorphic bundle $E_{\mathbb{C}}$ —the Grassmannian bundle, $Gr_n(E_{\mathbb{C}})$.

Example 2.19 (Pseudoholomorphic Curves). Let $E \longrightarrow (M, I)$ be an even dimensional vector bundle fibered over a complex curve. If ∇ is any connection on E then $R^{0,2}$ is automatically zero and hence $(\mathcal{C}(E), \mathcal{J}^{(\nabla, I)})$ is a complex manifold. Moreover if g is a metric on E and ∇ is a metric connection then $\mathcal{T}(E, g)$ is a complex submanifold of $(\mathcal{C}(E), \mathcal{J}^{(\nabla, I)})$.

As an application, let $E \longrightarrow (N, J)$ be an even dimensional vector bundle that is fibered over an almost complex manifold and let ∇ be any connection on E . The goal is to show that although $(\mathcal{C}(E), \mathcal{J}^{(\nabla, J)})$ is only an almost complex manifold, it always contains many pseudoholomorphic submanifolds that are in fact complex manifolds. The idea is to use the well known existence of a plethora of pseudoholomorphic curves in N . Indeed, if we let $i : (S, I) \longrightarrow (N, J)$ be a pseudoholomorphic embedding of a complex curve into N then the curvature of $i^*\nabla$ on i^*E is (1,1) and thus $(\mathcal{C}(i^*E), \mathcal{J}^{(i^*\nabla, I)})$ is a complex manifold. As it is straightforward to show that i induces a pseudoholomorphic embedding of $\mathcal{C}(i^*E)$ into $\mathcal{C}(E)$, $\mathcal{C}(i^*E)$ is one of many examples of pseudoholomorphic submanifolds of $\mathcal{C}(E)$ that are themselves complex manifolds.

Further connections between twistors and pseudoholomorphic curves will be explored in the near future. \square

Another example of a vector bundle that naturally admits connections with (1,1) curvature is a holomorphic Hermitian bundle. We will describe

this case in detail in Sections 3.1-3.3, but in the following we present examples where the base manifold M is a twistor space itself as well as an anti-selfdual four manifold.

Example 2.20. (Twistors) Let V be an even dimensional real vector space and, as in Section 2.1, let $\mathcal{C}(V)$ be its twistor space with complex structure $I_{\mathcal{C}}$. The bundle E that we will consider is the trivial bundle $\mathcal{C}(V) \times V \rightarrow \mathcal{C}(V)$. We could then choose the trivial connection d on E to define a complex structure on $\mathcal{C}(E)$ but let us modify it by using a certain section ϕ of $\text{End}E \rightarrow \mathcal{C}(V)$ that is defined by $\phi|_J = J$. (ϕ has appeared before in Sections 2.1.1 and 2.2 where we were discussing the twistor spaces associated to bundles and not just vector spaces.) The connection that we will then choose is $\nabla := d + \frac{1}{2}(d\phi)\phi$ since its curvature has the desired property:

Proposition 2.21. ∇ is a connection on E with $(1,1)$ curvature.

To prove this, we need the following lemma, which is a special case of Proposition 2.7.

Lemma 2.22. Let $A \in T_J\mathcal{C} = \{B \in \text{End}V \mid \{B, J\} = 0\}$. Then $d_A\phi = A$.

Proof of Proposition 2.21. The curvature R^∇ is given by $-\frac{1}{4}(d\phi \wedge d\phi)$. To show that it is $(1,1)$ first note that if $A \in T_J\mathcal{C} \subset \text{End}V$ then $I_{\mathcal{C}}A = JA$ and by the above lemma, $d_A\phi = A$. It then follows that for A and $B \in T_J\mathcal{C}$, $R^\nabla(JA, JB) = -\frac{1}{4}[JA, JB]$ and since A and B anticommute with J , this equals $-\frac{1}{4}[A, B] = R^\nabla(A, B)$. Thus R^∇ is $(1,1)$. \square

It then follows from the above proposition as well as Theorem 2.17 that $(\mathcal{C}(E), \mathcal{J}^{(\nabla, I_{\mathcal{C}})})$ is a complex manifold.

Of course, we could have replaced $\frac{1}{2}(d\phi)\phi$ in the definition of ∇ with, for example, just $d\phi$. The reason that we chose this specific term is that ∇ will then satisfy $\nabla\phi = 0$, which will be used especially in [13].

In Section 3.4 we will show how this example can be understood as part of a general procedure that produces new connections with $(1,1)$ curvature from holomorphic sections of twistor space. \square

Example 2.23 (Anti-selfdual Curvatures). Consider an even dimensional vector bundle $E \rightarrow (M, g)$ that is fibered over a four dimensional oriented Riemannian manifold. Since the manifold is oriented, the bundle of 2-forms, $\wedge^2 T^*$, splits into a direct sum of \wedge^+ and \wedge^- , the $+1$ and -1 eigenbundles of the Hodge star operator. To obtain a complex structure on $\mathcal{C}(E)$, suppose ∇ is a connection on E with anti-selfdual curvature, i.e. $R^\nabla \in \Gamma(\wedge^- \otimes \text{End}E)$. Moreover, suppose I is a complex structure on M that is compatible with g and that also induces the same orientation as that of the given one. The claim then is that R^∇ is automatically $(1,1)$ with respect to I . The reason is that it is well known that $\wedge^+ = \langle w \rangle \oplus (\wedge^{2,0} \oplus \wedge^{0,2})$ and $\wedge^- = \wedge_0^{1,1}$, where $w(\cdot, \cdot) = g(I\cdot, \cdot)$ and $\wedge_0^{1,1}$ is the orthogonal complement to $\langle w \rangle$ in $\wedge^{1,1}$. We thus have:

Corollary 2.24. $(\mathcal{C}(E), \mathcal{J}^{(\nabla, I)})$ is a complex manifold.

Let us now consider the following special case, which is more familiar in the literature. Let $E = TM$ and let (M, g, I) be an anti-selfdual Hermitian four manifold whose orientation is determined by I . As the curvature of the Levi Civita connection, R^∇ , lies in $\Gamma(\wedge^- \otimes \mathfrak{o}(TM, g))$, there are at least two integrable complex structures on \mathcal{T}^+ , which is the subbundle of \mathcal{T} whose elements induce the same orientation as that of I . The first of these complex structures is $\mathcal{J}^{(\nabla, I)}$, as it is integrable on all of \mathcal{T} , and the second is the tautological complex structure $\mathcal{J}_{taut}^\nabla$ as defined in Section 2.3 [3].

Given that \mathcal{T}^+ admits these two complex structures, it is natural to explore some bundles over this twistor space that admit connections with (1,1) curvature, for they in turn can be used to define complex structures on other associated twistor spaces. Letting $\pi : \mathcal{T}^+ \rightarrow M$ be the projection map, the first bundle that we will consider is π^*TM . The claim then is that there are at least two connections on this bundle that have (1,1) curvatures with respect to both $\mathcal{J}^{(\nabla, I)}$ and $\mathcal{J}_{taut}^\nabla$. The first of these connections is simply $\pi^*\nabla$ —its curvature is (1,1) because $R^\nabla \in \Gamma(\wedge^- \otimes \mathfrak{o}(TM, g))$ —while the other connection is $\pi^*\nabla' = \pi^*\nabla + \frac{1}{2}(\pi^*\nabla\phi)\phi$, where $\phi \in \Gamma(\pi^*EndTM)$ is defined by $\phi|_J = J$. One way to prove that this latter connection has the desired (1,1) curvature property is to generalize the proof of Proposition 2.21. Although this is straightforward to carry out, we will prove it instead in Section 3.4 by first showing that ϕ is a holomorphic section of $(\mathcal{C}(\pi^*TM), \mathcal{J}^{(\pi^*\nabla, \mathcal{I})}) \rightarrow (\mathcal{T}^+, \mathcal{I})$ for $\mathcal{I} \in \{\mathcal{J}^{(\nabla, I)}, \mathcal{J}_{taut}^\nabla\}$. Note that, similar to the discussion in the previous example, the connection $\pi^*\nabla'$ satisfies $\pi^*\nabla'\phi = 0$, which will have several applications in [13].

Now we can use these connections on π^*TM to produce connections on the tangent bundle of \mathcal{T}^+ , $T\mathcal{T}^+$, that also have (1,1) curvatures with respect to both $\mathcal{J}^{(\nabla, I)}$ and $\mathcal{J}_{taut}^\nabla$. For this, split $T\mathcal{T}^+ = V\mathcal{T}^+ \oplus H^\nabla\mathcal{T}^+$, which is a special case of Lemma 2.6, and identify $V\mathcal{T}^+$ with $[\pi^*(\mathfrak{o}(TM, g)), \phi]$ and $H^\nabla\mathcal{T}^+$ with π^*TM . As one may check, $\pi^*\nabla + \frac{1}{2}[(\pi^*\nabla\phi)\phi, \cdot] \oplus D$, where D is either $\pi^*\nabla$ or $\pi^*\nabla'$, defines a connection on $T\mathcal{T}^+$ with (1,1) curvature and thus defines different complex structures on $\mathcal{C}(T\mathcal{T}^+)$.

We will consider more properties of $\mathcal{J}^{(\nabla, I)}$ and $\mathcal{J}_{taut}^\nabla$ as well as their interaction in Section 3.4.2 and in [13]. \square

In Section 3, we will give more examples of bundles that admit connections with (1,1) curvature and show, in particular, that SKT, bihermitian as well as Calabi-Yau manifolds naturally admit complex structures on their twistor spaces.

2.5. Other Curvature Conditions. Although, by Theorem 2.16, the condition $R^{(0,2)} = 0$ guarantees the integrability of $\mathcal{J}^{(\nabla, I)}$ on $\mathcal{C}' \subset \mathcal{C}(E)$, it is not the most general one. The present goal is to demonstrate some of these more general conditions for certain \mathcal{C}' .

As a first example, consider a \mathcal{C}' that satisfies the following condition: given any $J \in \mathcal{C}'$, $-J$ is also in \mathcal{C}' .

Proposition 2.25. *If \mathcal{C}' satisfies the above condition then $(\mathcal{C}', \mathcal{J}^{(\nabla, I)})$ is a complex manifold if and only if $[R^{0,2}, J] = 0$ for all $J \in \mathcal{C}'$.*

Proof. If $(\mathcal{C}', \mathcal{J}^{(\nabla, I)})$ is a complex manifold then given $J \in \mathcal{C}'$, it follows from Theorem 2.16 that $[R^{0,2}, J]E_J^{0,1}$ and $[R^{0,2}, J]E_{-J}^{0,1}$ are both zero. Hence $[R^{0,2}, J] = 0$ for all $J \in \mathcal{C}'$. As I is already assumed to be integrable, the converse also follows from Theorem 2.16. \square

In the case when $\mathcal{C}' = \mathcal{C}$, it is straightforward to show that the condition $[R^{0,2}, J] = 0$ for all $J \in \mathcal{C}$ is equivalent to the endomorphism part of $R^{0,2}$ being pointwise constant. We thus have:

Proposition 2.26. *$(\mathcal{C}, \mathcal{J}^{(\nabla, I)})$ is a complex manifold if and only if $R^{(0,2)} = \lambda \otimes \mathbf{1}$, where λ is a $(0,2)$ form on M and $\mathbf{1}$ is the identity endomorphism on $E_{\mathbb{C}}$.*

Example 2.27. To take a simple example, let ∇' be a connection on $E \rightarrow (M, I)$ that has $(1,1)$ curvature and let $\nabla = \nabla' + (w \otimes \mathbf{1})$ for some 1-form w . Then $(R^{\nabla})^{0,2} = (\nabla^{0,1})^2$ on $E_{\mathbb{C}}$ equals $\bar{\partial}w^{0,1} \otimes \mathbf{1}$ and hence $\mathcal{J}^{(\nabla, I)}$ is a complex structure on \mathcal{C} . This complex structure, however, is not new since $\mathcal{J}^{(\nabla, I)}$ is actually equal to $\mathcal{J}^{(\nabla', I)}$. The reason is that although the connections ∇ and ∇' are not equal on E they are in fact the same on $\text{End}E$.

More interesting examples will be the subject of future work. \square

For another example of a \mathcal{C}' of the above type, let g be a metric on $E \rightarrow (M, I)$ and let ∇ be a metric connection. As in the case for \mathcal{C} , it follows from Proposition 2.25 that $\mathcal{J}^{(\nabla, I)}$ is integrable on $\mathcal{C}' = \mathcal{T}(g)$ if and only if the endomorphism part of $R^{0,2}$ is pointwise constant. However, in this case $R^{0,2}$ is a $(0,2)$ form that takes values in the skew endomorphism bundle $\mathfrak{o}(E_{\mathbb{C}}, g)$, so that its trace is zero. We thus have:

Proposition 2.28. *$(\mathcal{T}, \mathcal{J}^{(\nabla, I)})$ is a complex manifold if and only if $R^{(0,2)} = 0$.*

Example 2.29. In [13] we will be considering other types of \mathcal{C}' . For example, let $E \rightarrow (M, I)$ be equipped with a metric g and a metric connection ∇ . If $J \in \Gamma(\mathcal{T})$ satisfies $\nabla J = 0$ then we will show, in particular, that the following are almost complex submanifolds of $(\mathcal{T}, \mathcal{J}^{(\nabla, I)})$:

$$\begin{aligned} 1) \mathcal{T}^{(m_1, *)}(J) &= \{K \in \mathcal{T} \mid \dim \text{Ker}(K + J) = 2m_1\} \\ 2) \mathcal{T}^{(*, m-1)}(J) &= \{K \in \mathcal{T} \mid \dim \text{Ker}(K - J) = 2m_{-1}\}. \end{aligned}$$

In addition, if R^{∇} is $(1,1)$ then the above are complex submanifolds that form the strata of some of the stratifications of \mathcal{T} that were mentioned in the introduction.

So defined, these submanifolds are naturally fiber subbundles of \mathcal{T} and we will show in [13] that their fibers are related to certain Schubert cells. We will explore more properties of these subbundles in that paper. \square

3. MORE EXAMPLES

The goal of the next few sections is to describe various connections with (1,1) curvature on holomorphic Hermitian bundles and the resulting complex structures on the twistor spaces \mathcal{C} and \mathcal{T} . In particular, we will demonstrate that the twistor spaces of SKT, bihermitian and Calabi-Yau manifolds naturally admit complex structures. In Section 3.4, we will explore the holomorphic sections of these twistor spaces and show how they can be used to construct other connections that have (1,1) curvature.

We will now begin by considering the Chern connections of Hermitian bundles.

3.1. Chern Connections. Let $E \rightarrow (M, I)$ be a holomorphic bundle fibered over a complex manifold. Here, we will view it as a real bundle equipped with a fiberwise complex structure, J . If g is any fiberwise metric on E that is compatible with J then, as is well known, the associated Chern connection ∇^{Ch} (considered as a real connection on E) has (1,1) curvature. We thus have

Corollary 3.1. *$(\mathcal{C}, \mathcal{J}^{(\nabla^{Ch}, I)})$ is a complex manifold and \mathcal{T} is a complex submanifold.*

Example 3.2. As a simple example, let (M, I) be any complex manifold that admits a Kahler metric g . Then the Chern connection, ∇^{Ch} , on TM equals the Levi Civita connection, ∇^{Levi} . Thus $\mathcal{J}^{(\nabla^{Levi}, I)}$ is an integrable complex structure on \mathcal{C} and \mathcal{T} . \square

If we now \mathbb{C} -linearly extend ∇^{Ch} to $E_{\mathbb{C}}$ then, as a particular case of Lemma 2.18, $\nabla^{Ch(0,1)}$ is a $\bar{\partial}$ -operator for this bundle. To describe this $\bar{\partial}$ -operator in more familiar terms, let us consider the holomorphic bundle $E^{1,0} \oplus E^{*1,0}$, where $E^{1,0}$ is the $+i$ eigenbundle of J . The claim then is that the map

$$1 \oplus g : E_{\mathbb{C}} = E^{1,0} \oplus E^{0,1} \rightarrow E^{1,0} \oplus E^{*1,0}$$

is an isomorphism of holomorphic vector bundles. If we denote the Chern connection on $E^{1,0}$ by $\tilde{\nabla}^{Ch}$ then this follows from the following proposition, whose proof is straightforward.

Proposition 3.3. *$\nabla^{Ch} = \tilde{\nabla}^{Ch} \oplus g^{-1} \tilde{\nabla}^{Ch} g$, as complex connections on $E_{\mathbb{C}} = E^{1,0} \oplus E^{0,1}$.*

Thus in particular if $\{e_i\}$ is a local holomorphic trivialization of $E^{1,0}$ then $\{e_i, g^{-1}(e^i)\}$ is a holomorphic trivialization of $E_{\mathbb{C}}$.

Now if g' is another metric on E that is compatible with J then in Section 4 we will address the question of whether $(\mathcal{T}(g'), \mathcal{J}^{(\nabla^{Ch'}, I)})$ is biholomorphic to $(\mathcal{T}(g), \mathcal{J}^{(\nabla^{Ch}, I)})$ by holomorphically embedding twistor spaces into Grassmannian bundles.

3.2. $\bar{\partial}$ -operators. In the previous section, we found it useful to describe $\nabla^{Ch(0,1)}$ on $E_{\mathbb{C}}$ by considering the natural $\bar{\partial}$ -operator $\bar{\partial}$ on $E^{1,0} \oplus E^{*1,0}$ and the isomorphism

$$1 \oplus g : E_{\mathbb{C}} = E^{1,0} \oplus E^{0,1} \longrightarrow E^{1,0} \oplus E^{*1,0}.$$

In this section, we will give more examples of $\bar{\partial}$ -operators on $E^{1,0} \oplus E^{*1,0}$ and use this same isomorphism to transfer them to ones on $E_{\mathbb{C}}$. These in turn will give metric connections on E with (1,1) curvature that can be used to define complex structures on \mathcal{T} .

To begin, let $(E, g, J) \longrightarrow (M, I)$ be, as above, a holomorphic Hermitian vector bundle and consider the following natural symmetric bilinear form \langle, \rangle on $E^{1,0} \oplus E^{*1,0}$: $\langle X + \mu, Y + \nu \rangle = \frac{1}{2}(\mu(Y) + \nu(X))$. A general $\bar{\partial}$ -operator that preserves this metric is of the form $\bar{\partial} + \mathcal{D}^{0,1}$, where $\mathcal{D}^{0,1} \in \Gamma(T^{*0,1} \otimes \mathfrak{so}(E^{1,0} \oplus E^{*1,0}))$. If we now consider the splitting of $\mathfrak{so}(E^{1,0} \oplus E^{*1,0}) = \text{End}E^{1,0} \oplus \wedge^2 E^{*1,0} \oplus \wedge^2 E^{1,0}$ then we may decompose

$$\mathcal{D}^{0,1} = \begin{pmatrix} A & \alpha \\ D & -A^t \end{pmatrix},$$

where A, D and α are (0,1) forms with values in $\text{End}E^{1,0}, \wedge^2 E^{*1,0}$ and $\wedge^2 E^{1,0}$, respectively.

Since $\bar{\partial} + \mathcal{D}^{0,1}$ squares to zero, there are differential conditions on these sections. If we take, for example, the case when $\mathcal{D}^{0,1} = D$ then these conditions are equivalent to $\bar{\partial}D = 0$; a similar statement holds for the case when $\mathcal{D}^{0,1} = \alpha$.

To obtain $\bar{\partial}$ -operators on $E_{\mathbb{C}}$, consider, as above, the isomorphism,

$$1 \oplus g : (E_{\mathbb{C}} = E^{1,0} \oplus E^{0,1}, \frac{g}{2}) \longrightarrow (E^{1,0} \oplus E^{*1,0}, \langle, \rangle).$$

$\bar{\partial} + \mathcal{D}^{0,1}$ on $E^{1,0} \oplus E^{*1,0}$ then corresponds to $\nabla^{Ch(0,1)} + \mathcal{D}_g^{0,1}$ on $E_{\mathbb{C}}$, where

$$\mathcal{D}_g^{0,1} = \begin{pmatrix} A & \alpha g \\ g^{-1}D & -g^{-1}A^t g \end{pmatrix}.$$

As we are interested in real connections on E , note that $\nabla^{Ch(0,1)} + \mathcal{D}_g^{0,1}$ is the (0,1) part of the real connection $\nabla^{Ch} + \mathcal{D}_g := \nabla^{Ch} + \mathcal{D}_g^{0,1} + \overline{\mathcal{D}_g^{0,1}}$, whose curvature is (1,1).

Corollary 3.4. $\mathcal{J}^{(\nabla^{Ch} + \mathcal{D}_g, I)}$ is a complex structure on \mathcal{C} and \mathcal{T} .

For convenience, we summarize the $\bar{\partial}$ -operators and connections that we have discussed so far in the following table.

E	$E_{\mathbb{C}}$	$E^{1,0} \oplus E^{*1,0}$
$\nabla^{Ch} + \mathcal{D}_g$	$\nabla^{Ch(0,1)} + \mathcal{D}_g^{0,1}$	$\bar{\partial} + \mathcal{D}^{0,1}$

If we now take the case when $\mathcal{D}^{0,1} = D$ then in Section 4.4 we will explore how $\mathcal{J}^{(\nabla^{Ch} + \mathcal{D}_g, I)}$ on \mathcal{T} depends on the Dolbeault cohomology class of D in $H^{0,1}(\wedge^2 E^{*,1,0})$, i.e. if $B \in \Gamma(\wedge^2 E^{*,1,0})$ then we will determine whether $\bar{\partial} + D$ and $\bar{\partial} + D + \bar{\partial}B$ give isomorphic complex structures on \mathcal{T} .

Moreover we will also address a question that is a generalization of the one raised in the previous section: if g' were another metric on E that is compatible with J then given $\mathcal{D}^{0,1} \in \Gamma(T^{*,0,1} \otimes \mathfrak{so}(E^{1,0} \oplus E^{*,1,0}))$, is it true that $(\mathcal{T}(g'), \mathcal{J}^{(\nabla^{Ch'} + \mathcal{D}_{g'}, I)})$ is biholomorphic to $(\mathcal{T}(g), \mathcal{J}^{(\nabla^{Ch} + \mathcal{D}_g, I)})$?

3.3. Three Forms. An important case of the above discussion is when $E = TM$ is fibered over a Hermitian manifold (M, g, I) that is equipped with a real three form $H = \overline{H}^{2,1} + H^{2,1}$ of type $(1,2) + (2,1)$, such that $\bar{\partial}H^{2,1} = 0$. In this case, we will let $\mathcal{D}^{0,1} = H^{2,1}$, which is defined to be a section of $T^{*,0,1} \otimes \mathfrak{so}(T^{1,0} \oplus T^{*,1,0})$ by setting $H_v^{2,1}w = H^{2,1}(v, w, \cdot)$, for $v \in T^{0,1}$ and $w \in T^{1,0}$. It then follows that $\nabla^{Ch(0,1)} + g^{-1}H^{2,1}$, where here $g^{-1}H^{2,1} = g^{-1}H_{\frac{1}{2}(1+iI)}^{2,1}$, is a $\bar{\partial}$ -operator on $TM_{\mathbb{C}} = T^{1,0} \oplus T^{0,1}$. As the corresponding \mathcal{D}_g in the above table is $\frac{1}{2}I[g^{-1}H, I]$, we have

Proposition 3.5. $\nabla^{Ch} + \frac{1}{2}I[g^{-1}H, I]$ is a metric connection on TM with $(1,1)$ curvature.

Hence $\mathcal{J}^{(\nabla^{Ch} + \frac{1}{2}I[g^{-1}H, I], I)}$ is a complex structure on \mathcal{C} and \mathcal{T} .

As we will now show, natural examples of the above three form H can be found on SKT manifolds, bihermitian manifolds and Calabi-Yau threefolds.

3.3.1. SKT Manifolds. A natural example of a real three form on any Hermitian manifold, (M, g, I) , is $H = -d^c w = i(\partial - \bar{\partial})w$, where $w(\cdot, \cdot) = g(I\cdot, \cdot)$. If we take its $(2,1)$ part, $H^{2,1}$, then it is straightforward to check that it is $\bar{\partial}$ closed if and only if $dH = 0$. Manifolds whose H satisfy this condition are known in the literature as strong Kahler with torsion (SKT) manifolds and have recently become very popular in the mathematics and physics communities [10, 9]. One of the associated $\bar{\partial}$ -operators on $TM_{\mathbb{C}} = T^{1,0} \oplus T^{0,1}$ is $\nabla^{Ch(0,1)} - g^{-1}H^{2,1}$ and was actually introduced in a paper of Bismut in his study of Dirac operators [5]. The main point that we would like to stress here is that, as a corollary of the above discussion, this $\bar{\partial}$ -operator leads to complex structures on the twistor spaces \mathcal{C} and \mathcal{T} that can be described as follows. First note that $\nabla^{Ch(0,1)} - g^{-1}H^{2,1}$ is the $(0,1)$ part of the real connection $\nabla^{Ch} - \frac{1}{2}I[g^{-1}H, I]$ which can be shown to be equal to $\nabla^- := \nabla^{Levi} - \frac{1}{2}g^{-1}H$, where ∇^{Levi} is the Levi Civita connection. The connection ∇^- is closely related to the Bismut connection, $\nabla^+ := \nabla^{Levi} + \frac{1}{2}g^{-1}H$ (see below for a general definition as well as [5, 12]).

Corollary 3.6. If (M, g, I) is SKT then $(\mathcal{C}, \mathcal{J}^{(\nabla^-, I)})$ is a complex manifold and \mathcal{T} is a complex submanifold.

The Bismut connection that was mentioned above is actually defined for any almost Hermitian manifold, (M, g, I) :

Definition 3.7. The Bismut connection is the unique connection, ∇^+ , that satisfies

1. $\nabla^+ = \nabla^{Levi} + \frac{1}{2}g^{-1}H$, where H is a 3-form
2. $\nabla^+ I = 0$.

It can be shown that H is $(1,2) + (2,1)$ if and only if I is integrable and in this case it equals $-d^c w$ [12, 15].

3.3.2. Bihermitian Manifolds. A source of SKT manifolds is bihermitian manifolds. They were first introduced by physicists in [11], motivated by studying certain supersymmetric sigma models, and were later found to be equivalent to (twisted) generalized Kahler manifolds [15, 18] (see also [1]). A bihermitian manifold is by definition a Riemannian manifold (M, g) that is equipped with two metric compatible complex structures J_+ and J_- that satisfy the following conditions

$$\nabla^+ J_+ = 0 \quad \text{and} \quad \nabla^- J_- = 0,$$

where $\nabla^\pm = \nabla^{Levi} \pm \frac{1}{2}g^{-1}H$, for a closed three form H .

It then follows from Definition 3.7 that ∇^+ and ∇^- are the respective Bismut connections for J_+ and J_- . Thus an equivalent way to express the above bihermitian conditions is

$$H = -d_+^c w_+ = d_-^c w_- \quad \text{and} \quad dH = 0.$$

Since dH is assumed to be zero, (g, J_+) and (g, J_-) are two SKT structures for M and hence by Corollary 3.6 the associated twistor space \mathcal{T} admits the following two complex structures that depend on the three form H :

Corollary 3.8. $\mathcal{J}^{(\nabla^-, J_+)}$ and $\mathcal{J}^{(\nabla^+, J_-)}$ are two complex structures for \mathcal{C} and \mathcal{T} .

We will derive more results about bihermitian manifolds in Section 3.4.2 and in [13]. As for some examples, Kahler and hyperkahler manifolds are bihermitian. Other examples of bihermitian structures have been found, in particular, on compact even dimensional Lie groups, Del Pezzo surfaces and more generally on Fano manifolds [15, 19, 17].

3.3.3. Calabi-Yau Threefolds. Another class of Hermitian manifolds that admit $\bar{\partial}$ closed $(2,1)$ forms, which by Proposition 3.5 can be used to define complex structures on \mathcal{C} and \mathcal{T} , are Calabi-Yau threefolds. Indeed, $H_{Dolbeault}^{2,1}$ parametrizes the deformations of the complex structure on the threefold. It is interesting to note that at the same time there is a well defined map from $H_{Dolbeault}^{2,1}$ to the space of complex structures on the twistor space (modulo biholomorphisms) as will be described for a more general setup in Section 4.4.1. We are currently investigating the connection between deformations of complex structures on Calabi-Yau threefolds (as well as on general complex manifolds) and the complex geometries of twistor space.

3.4. Holomorphic Sections of Twistor space. In the previous sections we gave a number of examples of the general setup of a bundle $E \rightarrow (M, I)$ that is equipped with a connection, ∇ , that has (1,1) curvature. Given the associated complex manifold $(\mathcal{C}, \mathcal{J}^{(\nabla, I)})$, which holomorphically fibers over M , it is natural to consider its holomorphic sections. While we will use these sections in [13] to produce holomorphic subvarieties in M and in particular stratifications of \mathcal{C} , as was mentioned in the introduction, our focus here is to use them to construct more connections with (1,1) curvature— and thus more complex structures on twistor spaces. To begin, let us characterize the holomorphic sections of \mathcal{C} .

3.4.1. Holomorphic Sections and (1,1) Curvature. As above, let $E \rightarrow (M, I)$ be equipped with a connection ∇ that has (1,1) curvature.

Proposition 3.9. *$J : M \rightarrow (\mathcal{C}, \mathcal{J}^{(\nabla, I)})$ is a holomorphic section if and only if $J\nabla_v J = \nabla_{Iv} J$, for all $v \in TM$.*

Proof. Letting $P^\nabla : T\mathcal{C} \rightarrow V\mathcal{C}$, as in Section 2.2, be the projection operator that is based on the splitting of $T\mathcal{C}$ into $V\mathcal{C} \oplus H^\nabla\mathcal{C}$, let us consider the holomorphicity condition of J : $\mathcal{J}^{(\nabla, I)} J_* = J_* I$. If $v \in T_x M$, we then have:

(1) $\mathcal{J}^{(\nabla, I)} J_* v = \mathcal{J}^{(\nabla, I)} (P^\nabla(J_* v) + v^\nabla)$, where $v^\nabla \in H_{J(x)}^\nabla \mathcal{C}$ is the horizontal lift of $v \in T_x M$. This then equals $JP^\nabla(J_* v) + (Iv)^\nabla$.

(2) $J_*(Iv) = P^\nabla(J_* Iv) + (Iv)^\nabla$.

Hence J is holomorphic if and only if

$$(3.1) \quad JP^\nabla(J_* v) = P^\nabla(J_* Iv).$$

Using the formula $P^\nabla = \pi^* \nabla \phi$, as given in Proposition 2.7, it is straightforward to show that $P^\nabla(J_* v) = \nabla_v J$. Plugging this into Equation 3.1 proves Proposition 3.9. \square

If we consider the $\bar{\partial}$ -operator $\nabla^{0,1}$ on $E_{\mathbb{C}}$ then the above holomorphicity condition is equivalent to $(\nabla^{0,1} J)E_J^{0,1} = 0$. This in turn is equivalent to $J\nabla^{0,1} e = -i\nabla^{0,1} e$, for all $e \in \Gamma(E_J^{0,1})$. We thus have:

Proposition 3.10. *$J : M \rightarrow (\mathcal{C}, \mathcal{J}^{(\nabla, I)})$ is a holomorphic section if and only if $E_J^{0,1}$ is a holomorphic subbundle of $(E_{\mathbb{C}}, \nabla^{0,1})$.*

Having described the holomorphic sections of \mathcal{C} , let us now use them to build other connections on E with (1,1) curvature.

Proposition 3.11. *Let $J : M \rightarrow (\mathcal{C}, \mathcal{J}^{(\nabla, I)})$ be a holomorphic section. Then $\nabla + \nabla J(a + bJ)$, where $a, b \in \mathbb{R}$, is a connection with (1,1) curvature.*

Proof. We will show that $\nabla^{0,1} + \nabla^{0,1} J(a + bJ)$ is a $\bar{\partial}$ -operator on $E_{\mathbb{C}}$. By using Proposition 3.10, it is straightforward to show that this (0,1) connection is of the form $\nabla^{0,1} + A$, where $A \in \Gamma(T^{*0,1} \otimes \text{End} E_{\mathbb{C}})$ satisfies $\nabla^{0,1} A = 0$, $AE_J^{1,0} \subset E_J^{0,1}$ and $AE_J^{0,1} = 0$. It then follows that $\nabla^{0,1} + A$ squares to zero. \square

These new connections then lead to other complex structures on the twistor space:

Corollary 3.12. *If J is a holomorphic section of $(\mathcal{C}, \mathcal{J}^{(\nabla, I)})$ then $\mathcal{J}^{(\nabla + \nabla J(a+bJ), I)}$ is a complex structure on \mathcal{C} . In addition, if (E, g, J) is a Hermitian bundle and ∇ is a metric connection then this complex structure restricts to \mathcal{T} .*

Among the above connections, there is a particular one that we wish to focus on:

Proposition 3.13. $\nabla' := \nabla + \frac{1}{2}(\nabla J)J$ is a connection with $(1,1)$ curvature that satisfies $\nabla' J = 0$.

Thus J , originally chosen to be a holomorphic section of $(\mathcal{C}, \mathcal{J}^{(\nabla, I)})$, is also, by definition, a parallel section of $(\mathcal{C}, \mathcal{J}^{(\nabla', I)})$. Moreover, if we consider the holomorphic bundle $(E_{\mathbb{C}}, \nabla'^{0,1})$, then the fact that $\nabla' J = 0$ implies that $E_J^{1,0}$ and $E_J^{0,1}$ are holomorphic subbundles. We will explore the consequences of this in [13].

3.4.2. Examples of Holomorphic Sections. In searching for examples of holomorphic sections of $(\mathcal{C}, \mathcal{J}^{(\nabla, I)})$, it should first be noted that since $\pi : \mathcal{C} \rightarrow (M, I)$ is a holomorphic submersion, there are plenty of local ones. As for global holomorphic sections, in the case when $\mathcal{C} = \mathcal{C}(TM) \rightarrow (M, I)$, there is a natural candidate—namely I itself. While we will describe other examples of holomorphic sections later on, our first goal is to describe a certain class of twistor spaces, which include those associated to SKT and bihermitian manifolds, where I is holomorphic. Yet in fact we will find it natural to begin with a more general situation where I is not necessarily integrable and explore the condition that guarantees that I is pseudoholomorphic. We will use this, in particular, to show that the (pseudo)holomorphicity condition on sections of twistor space is a generalization of the integrability condition.

To begin, let (M, g, I) be an almost Hermitian manifold that is equipped with a real three form H . Using the natural Chern connection, ∇^{Ch} , on TM (see for example [12]) we will consider $\nabla = \nabla^{Ch} + \frac{1}{2}I[g^{-1}H, I]$ and the corresponding twistor space $(\mathcal{T}, \mathcal{J}^{(\nabla, I)})$, which is only an almost complex manifold. We will now explore the conditions on H so that I and $-I$ are pseudoholomorphic sections:

Proposition 3.14.

- 1) $I : M \rightarrow \mathcal{T}$ is pseudoholomorphic if and only if H is $(1,2) + (2,1)$.
- 2) $-I : M \rightarrow \mathcal{T}$ is pseudoholomorphic if and only if H is $(3,0) + (0,3)$.

Proof. As Proposition 3.9 is true regardless of whether I is integrable, it follows that I is pseudoholomorphic if and only if $I[g^{-1}H, I] = [g^{-1}H_I, I]$, which is equivalent to H being $(1,2) + (2,1)$. The proof of 2) is similar. \square

If we now choose H to be the three form that is contained in the Bismut connection $\nabla^+ = \nabla^{Levi} + \frac{1}{2}g^{-1}H$, which was defined in Definition 3.7, then the pseudoholomorphicity of I is equivalent to its integrability:

Proposition 3.15. $I : M \longrightarrow (\mathcal{T}, \mathcal{J}^{(\nabla, I)})$ is pseudoholomorphic if and only if I is integrable.

Proof. This follows from Proposition 3.14 and the fact that for the three form H in the Bismut connection, I is integrable if and only if H is $(1,2)+(2,1)$. \square

It is in this sense, as we remarked above, that the pseudoholomorphic condition on sections of twistor space generalizes the integrability condition on almost complex structures.

Using Proposition 3.14, we can now describe a class of holomorphic twistor spaces that always admit I as a holomorphic section. Indeed, let (M, g, I) be a Hermitian manifold that is equipped with a $\bar{\partial}$ closed $(2,1)$ form, $H^{2,1}$, and let $H = \overline{H^{2,1}} + H^{2,1}$. By Proposition 3.5, the connection $\nabla = \nabla^{Ch} + \frac{1}{2}I[g^{-1}H, I]$ on TM has $(1,1)$ curvature and thus $(\mathcal{T}, \mathcal{J}^{(\nabla, I)})$ is a complex manifold. Since H is $(1,2)+(2,1)$, we have:

Corollary 3.16. $I : M \longrightarrow (\mathcal{T}, \mathcal{J}^{(\nabla, I)})$ is holomorphic.

As was described in the previous section, whenever we have a holomorphic section of a twistor space we automatically obtain other connections with $(1,1)$ curvature. For the above case, one such connection is $\nabla' = \nabla + \frac{1}{2}(\nabla I)I$, which satisfies $\nabla' I = 0$. This particular connection is in fact a familiar one:

Proposition 3.17. $\nabla' = \nabla^{Ch}$

Proof. Since $\nabla I = [g^{-1}H, I]$, $\nabla' = \nabla^{Ch} + \frac{1}{2}I[g^{-1}H, I] + \frac{1}{2}([g^{-1}H, I])I = \nabla^{Ch}$. \square

To take an example of the above setup, let (M, g, I) be an SKT manifold, as defined in Section 3.3.1, so that $H = -d^c w = i(\partial - \bar{\partial})w$ satisfies $dH = 0$. Since this latter condition is equivalent to $\bar{\partial}H^{2,1} = 0$, the connection $\nabla^{Ch} - \frac{1}{2}I[g^{-1}H, I]$, which equals $\nabla^- = \nabla^{Levi} - \frac{1}{2}g^{-1}H$, has $(1,1)$ curvature. Taking Proposition 3.14 into account, we have:

Proposition 3.18. Let (M, g, I) be an SKT manifold.

- 1) $I : M \longrightarrow (\mathcal{T}, \mathcal{J}^{(\nabla^-, I)})$ is holomorphic.
- 2) $-I : M \longrightarrow (\mathcal{T}, \mathcal{J}^{(\nabla^-, I)})$ is holomorphic if and only if (M, g, I) is Kahler.

Proof. To prove 2), note that by Proposition 3.14, $-I$ is holomorphic if and only if H is $(3,0)+(0,3)$. As H is already $(1,2)+(2,1)$, this is true if and only if $H = 0$, or equivalently, (M, g, I) is Kahler. \square

Since a bihermitian manifold admits the two SKT structures (g, J_+) and (g, J_-) as explained in Section 3.3.2, we have

Proposition 3.19. *Let (M, g, J_+, J_-) be a bihermitian manifold. The following are holomorphic sections:*

$$\begin{aligned} 1) J_+ : (M, J_+) &\longrightarrow (\mathcal{T}, \mathcal{J}^{(\nabla^-, J_+)}) \\ 2) J_- : (M, J_-) &\longrightarrow (\mathcal{T}, \mathcal{J}^{(\nabla^+, J_-)}). \end{aligned}$$

Since $\nabla^\pm J_\pm = 0$, J_+ and J_- are also, by definition, parallel sections of $(\mathcal{T}, \mathcal{J}^{(\nabla^+, J_-)})$ and $(\mathcal{T}, \mathcal{J}^{(\nabla^-, J_+)})$ respectively. In [13] we will use these parallel and holomorphic sections of \mathcal{T} to produce holomorphic subvarieties in (M, J_+) and (M, J_-) that are new to the literature.

Having given some examples where the section $I : (M, g, I) \longrightarrow (\mathcal{T}, \mathcal{J}^{(\nabla, I)})$ is holomorphic, we will now consider other examples of holomorphic sections of twistor space. The following demonstrates that there are twistor spaces that are fibered over other twistor spaces that naturally admit holomorphic sections.

Example 3.20. Let $E \longrightarrow (M, I)$ be equipped with a connection ∇ that has (1,1) curvature. Denoting the projection map from $\mathcal{C}(E)$ to M by π , in this example we will be focusing on the complex manifold $(\mathcal{C}(E), \mathcal{I})$, where $\mathcal{I} = \mathcal{J}^{(\nabla, I)}$, along with its pullback bundle π^*E . Since π is holomorphic, the connection $\pi^*\nabla$ on π^*E has (1,1) curvature so that the total space of $(\mathcal{C}(\pi^*E), \mathcal{J}^{(\pi^*\nabla, \mathcal{I})}) \longrightarrow \mathcal{C}(E)$ is a complex manifold.

As we discussed before, ϕ is a natural section of $\mathcal{C}(\pi^*E)$, defined by $\phi|_J = J$, and the claim is that it is holomorphic:

Proposition 3.21. *ϕ is a holomorphic section of $(\mathcal{C}(\pi^*E), \mathcal{J}^{(\pi^*\nabla, \mathcal{I})}) \longrightarrow \mathcal{C}(E)$.*

Proof. It follows from Proposition 3.9 that ϕ is holomorphic if and only if $\phi(\pi^*\nabla)_X\phi = \pi^*\nabla_{\mathcal{I}X}\phi$, for all $X \in T\mathcal{C}(E)$. By Proposition 2.7, this is equivalent to $\phi P^\nabla(X) = P^\nabla(\mathcal{I}X)$, where $P^\nabla : T\mathcal{C} \longrightarrow V\mathcal{C}$ is the vertical projection operator that is induced by ∇ . This last expression follows directly from the definition of $\mathcal{I} = \mathcal{J}^{(\nabla, I)}$. \square

Since ϕ is holomorphic, by Proposition 3.13, the connection $\pi^*\nabla' = \pi^*\nabla + \frac{1}{2}(\pi^*\nabla\phi)\phi$ on $\pi^*E \longrightarrow (\mathcal{C}(E), \mathcal{I})$ also has (1,1) curvature and satisfies $\pi^*\nabla'\phi = 0$. We thus have:

Corollary 3.22. *$\mathcal{J}^{(\pi^*\nabla', \mathcal{I})}$ is another complex structure on $\mathcal{C}(\pi^*E)$.*

The fact that $R^{\pi^*\nabla'}$ is (1,1) implies that $(\pi^*E_\mathbb{C}, \pi^*\nabla'^{0,1})$ is a holomorphic bundle over $(\mathcal{C}(E), \mathcal{I})$ and since $\pi^*\nabla'\phi = 0$, $\pi^*E_\phi^{1,0}$ and $\pi^*E_\phi^{0,1}$ are holomorphic subbundles. We will use this result in [13] to produce complex submanifolds that form the strata of several stratifications of $(\mathcal{C}(E), \mathcal{I})$.

If we now restrict $(\pi^*E, \pi^*\nabla)$ to a particular fiber of $\pi : \mathcal{C}(E) \longrightarrow M$ then we obtain the setup of Example 2.20: an even dimensional real vector space, V , and the trivial bundle $E' = \mathcal{C}(V) \times V \longrightarrow \mathcal{C}(V)$ that is equipped with its

trivial connection d . It follows from Proposition 3.21 that $\phi \in \Gamma(\text{End}E')$, defined by $\phi|_J = J$, is a holomorphic section of $(\mathcal{C}(E'), \mathcal{J}^{(d, I_C)}) \rightarrow \mathcal{C}(V)$, where I_C is the standard complex structure on $\mathcal{C}(V)$. Another way to show the holomorphicity of ϕ is to first note that the map

$$\begin{aligned} (\mathcal{C}(E'), \mathcal{J}^{(d, I_C)}) &\rightarrow \mathcal{C}(V) \times \mathcal{C}(V) \\ J &\rightarrow (K, J), \end{aligned}$$

where $J \in \mathcal{C}(E')|_K$, is a biholomorphism. The section ϕ of $\mathcal{C}(E')$ then corresponds to the diagonal map from $\mathcal{C}(V)$ into $\mathcal{C}(V) \times \mathcal{C}(V)$, which is holomorphic.

We can now rederive the results of Example 2.20. (Note in that example E' was denoted by E .)

Corollary 3.23.

- 1) $\nabla' = d + \frac{1}{2}(d\phi)\phi$ has $(1,1)$ curvature.
- 2) $(\mathcal{C}(E'), \mathcal{J}^{(\nabla', \mathcal{I}_C)})$ is a complex manifold.

As an application of the above discussion, let $TM \rightarrow (M, I)$ be equipped with a connection ∇ of $(1,1)$ curvature and let $\pi : \mathcal{C} = \mathcal{C}(TM) \rightarrow M$ be the projection map. The goal is to prove a result that was stated in the introduction and in Section 2.3: that the almost complex structure $\mathcal{J}_{\text{taut}}^\nabla$ on \mathcal{C} , as defined in Definition 2.12, is a holomorphic section of $(\mathcal{C}(TC), \mathcal{J}) \rightarrow (\mathcal{C}, \mathcal{J}^{(\nabla, I)})$, for some appropriately chosen \mathcal{J} . To define \mathcal{J} , use ∇ to split $TC = VC \oplus H^\nabla C$ and identify VC with $[\text{End}(\pi^*TM), \phi]$ and $H^\nabla C$ with π^*TM . It then follows from the above discussion that $\tilde{\nabla} = \pi^*\nabla + \frac{1}{2}[(\pi^*\nabla\phi)\phi, \cdot] \oplus \pi^*\nabla$ is a connection on TC that has $(1,1)$ curvature with respect to $\mathcal{I} = \mathcal{J}^{(\nabla, I)}$. Hence $\mathcal{J} = \mathcal{J}^{(\tilde{\nabla}, \mathcal{I})}$ is a complex structure on $\mathcal{C}(TC)$. Using Proposition 3.21, it is then straightforward to show:

Proposition 3.24. $\mathcal{J}_{\text{taut}}^\nabla$ is a holomorphic section of $(\mathcal{C}(TC), \mathcal{J}^{(\tilde{\nabla}, \mathcal{I})}) \rightarrow (\mathcal{C}, \mathcal{I})$.

As another application, consider the setup in Example 2.23: a Hermitian anti-selfdual four manifold (M, g, I) whose orientation is determined by I , and $\pi : \mathcal{T}^+ \rightarrow M$, the subbundle of $\mathcal{T}(TM)$ whose elements induce the same orientation as the given one. If we let ∇ be the Levi Civita connection then as discussed in that example, $\pi^*\nabla$ is a connection on π^*TM that has $(1,1)$ curvature with respect to both of the integrable complex structures $\mathcal{J}^{(\nabla, I)}$ and $\mathcal{J}_{\text{taut}}^\nabla$ on \mathcal{T}^+ . Based on the discussion surrounding Proposition 3.21, it is straightforward to show that $\phi : (\mathcal{T}^+, \mathcal{I}) \rightarrow (\mathcal{C}(\pi^*TM), \mathcal{J}^{(\pi^*\nabla, \mathcal{I})})$ is not only holomorphic for $\mathcal{I} = \mathcal{J}^{(\nabla, I)}$ but for $\mathcal{I} = \mathcal{J}_{\text{taut}}^\nabla$ as well. Hence $\pi^*\nabla' = \pi^*\nabla + \frac{1}{2}(\pi^*\nabla\phi)\phi$ is a connection on π^*TM that has $(1,1)$ curvature with respect to both $\mathcal{J}^{(\nabla, I)}$ and $\mathcal{J}_{\text{taut}}^\nabla$, as was claimed in Example 2.23. As $\pi^*\nabla'\phi = 0$, we obtain several holomorphic structures on the bundles $\pi^*TM_\phi^{1,0}$ and $\pi^*TM_\phi^{0,1}$ that are fibered over $(\mathcal{T}^+, \mathcal{I})$. \square

More examples and applications of holomorphic sections of twistor spaces will be given in [13].

4. TWISTORS AND GRASSMANNIANS

In the previous sections, we have not only given examples of a bundle $E \rightarrow (M, I)$ with a connection ∇ that has (1,1) curvature but have also raised various questions about the complex manifold structure of $(\mathcal{C}, \mathcal{J}^{(\nabla, I)})$, especially in Section 3.2. In this section we will address these questions by holomorphically embedding \mathcal{C} into a more familiar complex manifold— a certain Grassmannian bundle. Indeed, as we noted previously, the condition that R^∇ is (1,1) is equivalent to $\nabla^{0,1}$ being a $\bar{\partial}$ -operator on $E_\mathbb{C}$, and if we let $\dim_\mathbb{R} E = 2n$ then the Grassmannian bundle that we will take will be the holomorphic bundle $Gr_n(E_\mathbb{C})$.

To define the embedding, we will first show how to holomorphically embed the fibers of \mathcal{C} into those of $Gr_n(E_\mathbb{C})$.

4.1. Embedding the Fibers. Let V be a $2n$ dimensional real vector space and let $Gr_n(V_\mathbb{C})$ be the Grassmannians of complex n planes. The map that we will consider is

$$\begin{aligned} \psi : \mathcal{C}(V) &\longrightarrow Gr_n(V_\mathbb{C}) \\ J &\longrightarrow V_J^{0,1}; \end{aligned}$$

it has the following properties:

Proposition 4.1.

1. The map $\psi : \mathcal{C}(V) \rightarrow Gr_n(V_\mathbb{C})$ is a holomorphic embedding.
2. The image of this embedding is $\{P \in Gr_n(V_\mathbb{C}) \mid P \oplus \bar{P} = V_\mathbb{C}\}$, which is an open submanifold of the Grassmannians.

Proof. Consider $\psi_* : T_J \mathcal{C}(V) \rightarrow T_{V_J^{0,1}} Gr_n(V_\mathbb{C})$ and choose the holomorphic chart

$$\begin{aligned} End(V_J^{0,1}, V_J^{1,0}) &\longrightarrow Gr_n(V_\mathbb{C}) \\ B &\longrightarrow Graph(B), \end{aligned}$$

where $Graph(B) = \{v^{0,1} + Bv^{0,1} \mid v^{0,1} \in V_J^{0,1}\}$. If we let A be a general element in $T_J \mathcal{C}(V) \cong \{D \in End V \mid \{D, J\} = 0\}$ then we need to show that $\psi_*(JA) = \mathcal{I}\psi_*(A)$, where \mathcal{I} is the complex structure on the Grassmannians.

First consider,

$$\begin{aligned} \psi_*(JA) &= \frac{d}{dt} \Big|_{t=0} \psi(\exp(\frac{-tA}{2}) J \exp(\frac{tA}{2})) \\ &= \frac{d}{dt} \Big|_{t=0} \exp(\frac{-tA}{2})(V_J^{0,1}). \end{aligned}$$

Using the above chart, $\psi_*(JA)$ then corresponds to $-\frac{A}{2}$, as an element of $End(V_J^{0,1}, V_J^{1,0})$.

Similarly we have $\psi_*(A) = \frac{d}{dt}|_{t=0} \exp(-\frac{tAJ}{2})(V_J^{0,1})$, so that under the above chart, $\mathcal{I}\psi_*(A)$ corresponds to $-\frac{iAJ}{2}$, which as an element of $\text{End}(V_J^{0,1}, V_J^{1,0})$ equals $-\frac{A}{2}$.

The proof of the other parts of the proposition are straightforward. \square

If we now choose a positive definite metric, g , on V then by restriction, the above map, ψ , gives a holomorphic embedding of $\mathcal{T}(V)$ into $Gr_n(V_{\mathbb{C}})$. Since the metric is positive definite, the image of this map is precisely $MI(V_{\mathbb{C}}) = \{P \in Gr_n(V_{\mathbb{C}}) | g(v, w) = 0, \forall v, w \in P\}$, the space of maximal isotropics of $V_{\mathbb{C}}$ defined by using the \mathbb{C} -bilinearly extended metric. For convenience we state this as a proposition.

Proposition 4.2.

$$\begin{aligned} \mathcal{T}(V) &\longrightarrow Gr_n(V_{\mathbb{C}}) \\ J &\longrightarrow V_J^{0,1} \end{aligned}$$

is a holomorphic embedding with image $MI(V_{\mathbb{C}})$.

4.2. The Holomorphic Embedding. Let us now consider a $2n$ dimensional real vector bundle $E \longrightarrow (M, I)$ that is fibered over a complex manifold. As discussed above, a connection ∇ on E with $(1,1)$ curvature gives rise to two complex analytic manifolds: the twistor space $(\mathcal{C}, \mathcal{J}^{(\nabla, I)})$ and the holomorphic fiber bundle $\pi_{Gr} : Gr_n(E_{\mathbb{C}}) \longrightarrow M$. To holomorphically embed \mathcal{C} into $Gr_n(E_{\mathbb{C}})$, we will generalize the map ψ that was defined in the previous section:

Theorem 4.3. *The map*

$$\begin{aligned} \psi : (\mathcal{C}, \mathcal{J}^{(\nabla, I)}) &\longrightarrow Gr_n(E_{\mathbb{C}}) \\ J &\longrightarrow E_J^{0,1} \end{aligned}$$

is a holomorphic embedding.

In the case when E is equipped with a metric g and ∇ is a metric connection, we will define $MI(E_{\mathbb{C}})$ to be the space of maximal isotropics in $Gr_n(E_{\mathbb{C}})$; we then have:

Proposition 4.4.

$$\begin{aligned} (\mathcal{T}, \mathcal{J}^{(\nabla, I)}) &\longrightarrow Gr_n(E_{\mathbb{C}}) \\ J &\longrightarrow E_J^{0,1} \end{aligned}$$

is a holomorphic embedding with image $MI(E_{\mathbb{C}})$.

To prove Theorem 4.3, we will need to describe the complex structure on the Grassmannians similarly to how we defined $\mathcal{J}^{(\nabla, I)}$ on \mathcal{C} . The first step will be to define the horizontal distribution $H^{\nabla}Gr_n$ on $Gr_n(E_{\mathbb{C}})$. But before giving the definition, let us first recall that if $P \in Gr_n(E_{\mathbb{C}})$ and $\gamma : \mathbb{R} \longrightarrow M$ satisfies $\gamma(0) = \pi_{Gr}(P)$ then we can use ∇ , considered as

a complex connection on $E_{\mathbb{C}}$, to parallel transport P along γ as follows. If we set $P = \langle e_1, \dots, e_n \rangle_{\mathbb{C}}$, so that $\{e_i\}$ is a basis for P , then define $P(t) = \langle e_1(t), \dots, e_n(t) \rangle_{\mathbb{C}}$, where $\gamma^* \nabla e_i(t) = 0$ and $e_i(0) = e_i$. Since ∇ is a complex connection on $E_{\mathbb{C}}$, $P(t)$ does not depend on the basis $\{e_i\}$ for P that was chosen.

With this, let us define the desired horizontal distribution on $Gr_n(E_{\mathbb{C}})$.

Definition 4.5. Let $H_P^{\nabla} Gr_n = \{ \frac{dP(t)}{dt} |_{t=0} | P(t) \text{ is the parallel translate of } P \text{ along } \gamma, \gamma(0) = \pi_{Gr}(P) \}$.

Along with $H^{\nabla} Gr_n$, there is also the natural vertical distribution VGr_n ; as it is defined by the fibers of $Gr_n(E_{\mathbb{C}})$, it is a complex vector bundle and satisfies $\pi_{Gr*}(V_P Gr_n) = 0$, for all $P \in Gr_n(E_{\mathbb{C}})$. It is straightforward to prove that these two distributions are complements to each other:

Lemma 4.6. $T_P Gr_n = V_P Gr_n \oplus H_P^{\nabla} Gr_n$.

We may now use the above lemma to define an almost complex structure on $Gr_n(E_{\mathbb{C}})$, which we will show in Proposition 4.8 to be the complex structure that is induced by $\nabla^{0,1}$ and which we will use to prove Theorem 4.3. As the definition of this almost complex structure is similar to that of $\mathcal{J}^{(\nabla, I)}$ on \mathcal{C} , we will denote it by the same symbol:

Definition 4.7. Let $\mathcal{J}^{(\nabla, I)}$ on $Gr_n(E_{\mathbb{C}})$ be defined as follows. First split

$$TGr_n = VGr_n \oplus H^{\nabla} Gr_n$$

and then let

$$\mathcal{J}^{(\nabla, I)} = \mathcal{J}^V \oplus \pi_{Gr}^* I,$$

where \mathcal{J}^V is the standard fiberwise complex structure on VGr_n and where we have used the natural identification of $H^{\nabla} Gr_n$ with $\pi_{Gr}^* TM$.

If we consider the complex manifold structure of $Gr_n(E_{\mathbb{C}})$ that is induced by the $\bar{\partial}$ -operator $\nabla^{0,1}$ on $E_{\mathbb{C}}$, we then have:

Proposition 4.8. *The complex structure on $Gr_n(E_{\mathbb{C}})$ is $\mathcal{J}^{(\nabla, I)}$.*

We will prove the above proposition for a more general setup in the next section; here we will use it to prove Theorem 4.3 by showing that the map $\psi : (\mathcal{C}, \mathcal{J}^{(\nabla, I)}) \rightarrow (Gr_n(E_{\mathbb{C}}), \mathcal{J}^{(\nabla, I)})$, which is given by $\psi(J) = E_J^{0,1}$, is holomorphic. Recalling the splitting of $T\mathcal{C} = V\mathcal{C} \oplus H^{\nabla} \mathcal{C}$, as given in Lemma 2.5, let us first consider the following:

Lemma 4.9. *The map ψ_* preserves horizontals: $\psi_* : H_J^{\nabla} \mathcal{C} \rightarrow H_{E_J^{0,1}}^{\nabla} Gr_n$.*

In fact, $\psi_(v^{\nabla}) = v^{(\nabla, Gr)}$, where v^{∇} and $v^{(\nabla, Gr)}$ are the appropriate horizontal lifts of $v \in T_x M$.*

Proof. Let $\gamma(t)$ be a curve in M such that $\gamma(0) = x$ and $\gamma'(0) = v$. Also let $J(t)$ be the parallel translate of $J \in \mathcal{C}(E_x)$ along γ (by using ∇), so that

$$\psi_*(v^{\nabla}) = \frac{d}{dt} |_{t=0} \psi(J(t)).$$

The claim then is that $\psi(J(t))$, which is by definition $E_{J(t)}^{0,1}$, equals $E_J^{0,1}(t)$, the parallel translate of $E_J^{0,1}$ along γ . To show this just note that if $e(t)$ is the parallel translate of $e \in E_J^{0,1}$ then $J(t)e(t)$ is also parallel and since $Je = -ie$, it follows that $J(t)e(t) = -ie(t)$ for all relevant $t \in \mathbb{R}$. Hence $\frac{d}{dt}|_{t=0}\psi(J(t)) = \frac{d}{dt}|_{t=0}E_J^{0,1}(t) = v^{(\nabla, Gr)}$. \square

Assuming Proposition 4.8, we can now prove that ψ is holomorphic:

Proof of Theorem 4.3. Consider $\psi_* : T_J\mathcal{C} \rightarrow T_{E_J^{0,1}}Gr_n$. By Proposition 4.8, we need to show that $\psi_*\mathcal{J}^{(\nabla, I)} = \mathcal{J}^{(\nabla, I)}\psi_*$.

A) If $A \in V_J\mathcal{C}$, the vertical tangent space to J , then it follows from Proposition 4.1 that

$$\psi_*(JA) = \mathcal{J}^{(\nabla, I)}\psi_*(A),$$

so that ψ_* is holomorphic in the vertical directions.

B) As for the horizontal directions, let $v^\nabla \in H_J^\nabla\mathcal{C}$ be the horizontal lift of $v \in T_xM$. Then $\psi_*(\mathcal{J}^{(\nabla, I)}v^\nabla) = \psi_*((Iv)^\nabla)$, which by Lemma 4.9 equals $(Iv)^{(\nabla, Gr)}$. This in turn equals $\mathcal{J}^{(\nabla, I)}v^{(\nabla, Gr)} = \mathcal{J}^{(\nabla, I)}\psi_*(v^\nabla)$. \square

4.3. Proof of Proposition 4.8. In this section, we will prove a slightly more general version of Proposition 4.8; this will then complete the proof of Theorem 4.3. To begin, we will find it useful to describe the complex structures on holomorphic vector bundles:

Let $\pi_F : F \rightarrow (M, I)$ be a complex vector bundle that is equipped with a $\bar{\partial}$ -operator, $\bar{\partial}$, and let ∇ be a complex connection on F such that $\nabla^{0,1} = \bar{\partial}$. Below we will let $\mathcal{J}^{(\nabla, I)}$ be the almost complex structure on either F or $Gr_k(F)$ that is defined in a by now familiar way: use ∇ to split the appropriate tangent bundle into vertical and horizontal distributions, and define $\mathcal{J}^{(\nabla, I)}$ to be the direct sum of the given fiberwise complex structure on the verticals and the lift of I on the horizontals.

Proposition 4.10. *Let ∇ be a complex connection on F such that $\nabla^{0,1} = \bar{\partial}$. Then the associated complex structure on F is $\mathcal{J}^{(\nabla, I)}$.*

Proof. Let $\{f_i\}$ ($1 \leq i \leq \dim_{\mathbb{C}} F$) be a holomorphic frame for F over $U \subset M$ and let W be the complex vector space that is generated by $\{w_i\}$ over \mathbb{C} . To prove the proposition, we need to show that the map

$$\begin{aligned} \sigma : (F|_U, \mathcal{J}^{(\nabla, I)}) &\rightarrow U \times W \\ a_i f_i|_x &\rightarrow (x, a_i w_i) \end{aligned}$$

is holomorphic. For this, consider $\sigma_* : T_f F \rightarrow T_{\sigma(f)}(U \times W)$, where $\pi_F(f) = x$.

1) Since $\sigma|_x$ is a complex linear isomorphism from $F|_x$ to W , σ is holomorphic in the vertical directions, i.e., $\sigma_*(if') = i\sigma_*(f')$, where $f' \in V_f F = F|_x$.

2) As for the horizontal directions, we need to show that $\sigma_*(\mathcal{J}^{(\nabla, I)}v^\nabla) = \mathcal{I}\sigma_*(v^\nabla)$, where v^∇ is the horizontal lift of $v \in T_x M$ to $H_f^\nabla F \subset T_f F$ and \mathcal{I}

is the complex structure on $U \times W$. Let us first consider,

$$\begin{aligned}\sigma_*(\mathcal{J}^{(\nabla, I)}v^\nabla) &= \sigma_*((Iv)^\nabla) \\ &= \frac{d}{dt}|_{t=0}\sigma(f(t)),\end{aligned}$$

where $f(t)$ is the parallel translate of f along the curve $\gamma : \mathbb{R} \rightarrow M$ that satisfies $\gamma(0) = x$ and $\gamma'(0) = Iv$. If we let $f(t) = a_j(t)f_j|_{\gamma(t)}$ then the above equals

$$(Iv, \frac{da_j(t)}{dt}|_{t=0}w_j).$$

Similarly, $\sigma_*(v^\nabla) = (v, \frac{d\tilde{a}_j(t)}{dt}|_{t=0}w_j)$, where $\tilde{f}(t) = \tilde{a}_j(t)f_j|_{\tilde{\gamma}(t)}$ is the parallel translate of f along the curve $\tilde{\gamma} : \mathbb{R} \rightarrow M$ that satisfies $\tilde{\gamma}(0) = x$, $\tilde{\gamma}'(0) = v$. Now since $\mathcal{I}\sigma_*(v^\nabla) = (Iv, i\frac{d\tilde{a}_j(t)}{dt}|_{t=0}w_j)$, σ is holomorphic if and only if $\frac{da_j(t)}{dt}|_{t=0} = i\frac{d\tilde{a}_j(t)}{dt}|_{t=0}$.

To show this equality, note that the condition $\tilde{\gamma}^*\nabla\tilde{f}(t) = 0$ together with $a_j := \tilde{a}_j(0) = a_j(0)$ imply that $i\frac{d\tilde{a}_j(t)}{dt}|_{t=0}f_j = -ia_j\nabla_v f_j$. This then equals $-a_j\nabla_{Iv}f_j$ because $\nabla^{0,1}f_j = 0$, which in turn equals $\frac{da_j(t)}{dt}|_{t=0}f_j$ since $\gamma^*\nabla f(t) = 0$. Hence σ is holomorphic. \square

As for the Grassmannians, we have:

Proposition 4.11. *The complex structure on $Gr_k(F)$ that is induced from $(F, \bar{\partial})$ is $\mathcal{J}^{(\nabla, I)}$.*

The proof of the above proposition and hence of Proposition 4.8 is just a straightforward generalization of the previous proof. This then completes the proof of Theorem 4.3 as well.

4.4. Corollaries of the Embedding. We will now demonstrate some of the corollaries of the holomorphic embedding $\psi : (\mathcal{C}, \mathcal{J}^{(\nabla, I)}) \rightarrow Gr_n(E_{\mathbb{C}})$, as given in Theorem 4.3. In particular, we will address certain issues regarding the holomorphic structure of twistor spaces that were raised in Section 3.2.

Let E and E' be two real vector bundles of even dimension that are fibered over (M, I) and that are respectively equipped with connections ∇ and ∇' of (1,1) curvature.

Proposition 4.12. *Let $A : E \rightarrow E'$ be a bundle map such that its \mathbb{C} -extension, $A : (E_{\mathbb{C}}, \nabla^{0,1}) \rightarrow (E'_{\mathbb{C}}, \nabla'^{0,1})$ is an isomorphism of holomorphic vector bundles. Then this map induces a fiber preserving biholomorphism between $(\mathcal{C}(E), \mathcal{J}^{(\nabla, I)})$ and $(\mathcal{C}(E'), \mathcal{J}^{(\nabla', I)})$.*

Proof. The isomorphism $A : (E_{\mathbb{C}}, \nabla^{0,1}) \rightarrow (E'_{\mathbb{C}}, \nabla'^{0,1})$ induces the biholomorphism $\tilde{A} : Gr_n(E_{\mathbb{C}}) \rightarrow Gr_n(E'_{\mathbb{C}})$ that is defined by $\tilde{A}(< e_1, \dots, e_n >_{\mathbb{C}}) = < Ae_1, \dots, Ae_n >_{\mathbb{C}}$. Since A is a real map, \tilde{A} restricts to a biholomorphism between the set $\{P \in Gr_n(E_{\mathbb{C}}) | P \oplus \bar{P} = E_{\mathbb{C}}|_{\pi_{Gr}(P)}\}$ in $Gr_n(E_{\mathbb{C}})$ and

the corresponding one in $Gr_n(E'_\mathbb{C})$. The proposition then follows from Theorem 4.3 and Proposition 4.1, which show that these sets are respectively biholomorphic to $(\mathcal{C}(E), \mathcal{J}^{(\nabla, I)})$ and $(\mathcal{C}(E'), \mathcal{J}^{(\nabla', I)})$. \square

Now suppose that E and E' are also equipped with respective metrics g and g' and that the above connections preserve the appropriate metrics. If we \mathbb{C} -bilinearly extend g and g' to $E_\mathbb{C}$ and $E'_\mathbb{C}$, we then have

Proposition 4.13. *Let $A : (E_\mathbb{C}, \nabla^{0,1}) \longrightarrow (E'_\mathbb{C}, \nabla'^{0,1})$ be an isomorphism of holomorphic vector bundles that is orthogonal with respect to g and g' . Then A induces a fiber preserving biholomorphism between $(\mathcal{T}(E, g), \mathcal{J}^{(\nabla, I)})$ and $(\mathcal{T}(E', g'), \mathcal{J}^{(\nabla', I)})$.*

Proof. Similar to the proof of Proposition 4.12, the isomorphism $A : (E_\mathbb{C}, \nabla^{0,1}) \longrightarrow (E'_\mathbb{C}, \nabla'^{0,1})$ induces a biholomorphism $\tilde{A} : Gr_n(E_\mathbb{C}) \longrightarrow Gr_n(E'_\mathbb{C})$. Since A is an orthogonal map, \tilde{A} maps the space of maximal isotropics, $MI(E_\mathbb{C})$, in $Gr_n(E_\mathbb{C})$ to the one in $Gr_n(E'_\mathbb{C})$. The proposition then follows from Proposition 4.4, which shows that $(\mathcal{T}(E, g), \mathcal{J}^{(\nabla, I)})$ and $(\mathcal{T}(E', g'), \mathcal{J}^{(\nabla', I)})$ are respectively biholomorphic to $MI(E_\mathbb{C})$ and $MI(E'_\mathbb{C})$. \square

In the following two sections we consider some applications of the above propositions.

4.4.1. Cohomology Independence. Let $(E, g, J) \longrightarrow (M, I)$ be a holomorphic Hermitian bundle fibered over a complex manifold and let $\bar{\partial}$ be the standard $\bar{\partial}$ -operator on $E^{1,0} \oplus E^{*1,0}$, where $E^{1,0}$ is the $+i$ eigenbundle of J . If we choose $D \in \Gamma(T^{*0,1} \otimes \wedge^2 E^{*1,0})$ to satisfy $\bar{\partial}D = 0$ then, as described in Section 3.2, $\nabla^{Ch(0,1)} + g^{-1}D$ is a $\bar{\partial}$ -operator on $E_\mathbb{C} = E^{1,0} \oplus E^{0,1}$ and, for $\nabla = \nabla^{Ch} + g^{-1}D + \overline{g^{-1}D}$, the twistor space $(\mathcal{T}(E), \mathcal{J}^{(\nabla, I)})$ is a complex manifold. If we now let $B \in \Gamma(\wedge^2 E^{*1,0})$ then $\nabla^{Ch(0,1)} + g^{-1}(D + \bar{\partial}B)$ is another $\bar{\partial}$ -operator on $E_\mathbb{C}$ and it is natural to wonder, as in Section 3.2, whether the associated twistor space is biholomorphic to the previous one. In other words, does the above give a well defined mapping from the Dolbeault cohomology group $H^{0,1}(\wedge^2 E^{*1,0})$ to the isomorphism classes of complex structures on \mathcal{T} ?

By using Proposition 4.13, we will show here that such a mapping does indeed exist. As a first step, let us consider the section of $O(E_\mathbb{C}, g) \exp(g^{-1}B)$, which equals $(1 + g^{-1}B)$ since $(g^{-1}B)^2 = 0$. We then have

Proposition 4.14. *The map $\exp(-g^{-1}B) : (E_\mathbb{C}, \nabla^{Ch(0,1)} + g^{-1}D) \longrightarrow (E_\mathbb{C}, \nabla^{Ch(0,1)} + g^{-1}(D + \bar{\partial}B))$ is an isomorphism of holomorphic vector bundles.*

Proof. Let $(\nabla^{Ch(0,1)} + g^{-1}D)v = 0$ and consider

$$\begin{aligned} & (\nabla^{Ch(0,1)} + g^{-1}(D + \bar{\partial}B))(1 - g^{-1}B)v \\ &= -\nabla^{Ch(0,1)}(g^{-1}Bv) + (g^{-1}\bar{\partial}B)v \\ &= -(\nabla^{Ch(0,1)}g^{-1}B)v - g^{-1}B\nabla^{Ch(0,1)}v + (g^{-1}\bar{\partial}B)v. \end{aligned}$$

Since the first and last terms cancel, we are left with $-g^{-1}B\nabla^{Ch(0,1)}v = -g^{-1}B(-g^{-1}Dv) = 0$. This then proves the proposition. \square

By Proposition 4.13, we can now conclude that the twistor spaces mentioned above are biholomorphic:

Proposition 4.15. *$\exp(-g^{-1}B)$ induces a fiber preserving biholomorphism between $(\mathcal{T}, \mathcal{J}^{(\nabla, I)})$ and $(\mathcal{T}, \mathcal{J}^{(\nabla', I)})$, where $\nabla^{0,1} = \nabla^{Ch(0,1)} + g^{-1}D$ and $\nabla'^{0,1} = \nabla^{Ch(0,1)} + g^{-1}(D + \bar{\partial}B)$.*

As a corollary, we have

Proposition 4.16. *The map $[D] \rightarrow [\mathcal{J}^{(\nabla, I)}]$, where $\nabla^{0,1} = \nabla^{Ch(0,1)} + g^{-1}D$, from the Dolbeault cohomology group $H^{0,1}(\wedge^2 E^{*1,0})$ to the isomorphism classes of complex structures on $\mathcal{T}(E, g)$ is well defined.*

4.4.2. Changing the Metric. In the previous example we worked with a fixed metric g ; but what if we were to choose another metric g' on E that is compatible with J —then is it true that $(\mathcal{T}(g), \mathcal{J}^{(\nabla^{Ch}, I)})$ and $(\mathcal{T}(g'), \mathcal{J}^{(\nabla^{Ch'}, I)})$ are biholomorphic? This is part of a more general question that was posed in Section 3.2: in that section we used a fixed metric, g , to define $\bar{\partial}$ -operators on $E_{\mathbb{C}}$ and thus complex structures on $\mathcal{T}(g)$ —but if we were to choose another metric g' then do we obtain new complex manifolds by considering $\mathcal{T}(g')$?

To address these questions, let us first recall some of the details of that section. Let $(E, J) \rightarrow (M, I)$ be a holomorphic vector bundle, considered as a real bundle with fiberwise complex structure J , that is fibered over a complex manifold. Defining \langle, \rangle and $\bar{\partial}$ to be the standard inner product and $\bar{\partial}$ -operator on $E^{1,0} \oplus E^{*1,0}$, let us consider the $\bar{\partial}$ -operator $\bar{\partial} + \mathcal{D}^{0,1}$, where $\mathcal{D}^{0,1} \in \Gamma(T^{*0,1} \otimes \mathfrak{so}(E^{1,0} \oplus E^{*1,0}))$. If g is a metric on E that is compatible with J then, as in Section 3.2, we can use the orthogonal isomorphism

$$1 \oplus g : (E_{\mathbb{C}} = E^{1,0} \oplus E^{0,1}, \frac{g}{2}) \rightarrow (E^{1,0} \oplus E^{*1,0}, \langle, \rangle)$$

to obtain the $\bar{\partial}$ -operator $\nabla^{Ch(0,1)} + \mathcal{D}_g^{0,1}$ on $E_{\mathbb{C}}$ as well as the complex structure $\mathcal{J}^{(\nabla^{Ch} + \mathcal{D}_g, I)}$ on $\mathcal{T}(g)$. (Here, $\mathcal{D}_g = \mathcal{D}_g^{0,1} + \overline{\mathcal{D}_g^{0,1}}$.)

Similarly, if g' is another metric that is compatible with J then we have the complex structure $\mathcal{J}^{(\nabla^{Ch'} + \mathcal{D}_{g'}, I)}$ on $\mathcal{T}(g')$. The goal then is to use Proposition 4.13 to show that the complex manifolds $\mathcal{T}(g)$ and $\mathcal{T}(g')$ are equivalent under a fiberwise biholomorphism.

First note, that if we compose the map $(1 \oplus g)$ with $(1 \oplus g')^{-1}$ then we obtain the following isomorphism of holomorphic vector bundles:

$$\begin{aligned} (E_{\mathbb{C}}, \nabla^{Ch(0,1)} + \mathcal{D}_g^{0,1}) &\rightarrow (E_{\mathbb{C}}, \nabla^{Ch'(0,1)} + \mathcal{D}_{g'}^{0,1}) \\ v^{1,0} + v^{0,1} &\rightarrow (v^{1,0} + g'^{-1}gv^{0,1}), \end{aligned}$$

where we have used the decomposition, $E_{\mathbb{C}} = E^{1,0} \oplus E^{0,1}$. As this is an orthogonal map from $(E_{\mathbb{C}}, g)$ to $(E_{\mathbb{C}}, g')$, by Proposition 4.13 we have

Proposition 4.17. *There exists a fiber preserving biholomorphism between $(\mathcal{T}(g), \mathcal{J}^{(\nabla^{Ch} + \mathcal{D}_g, I)})$ and $(\mathcal{T}(g'), \mathcal{J}^{(\nabla^{Ch'} + \mathcal{D}_{g'}, I)})$.*

In particular, if we set $\mathcal{D}'^{(0,1)}$ to zero, we have:

Proposition 4.18. *Let $(E, J) \rightarrow (M, I)$ be a holomorphic vector bundle that is equipped with two Hermitian metrics g and g' . Then $(\mathcal{T}(g), \mathcal{J}^{(\nabla^{Ch}, I)})$ and $(\mathcal{T}(g'), \mathcal{J}^{(\nabla^{Ch'}, I)})$ are biholomorphic.*

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